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**TWISTED INTERTWINING OPERATORS, TENSOR  
PRODUCTS OF TWISTED MODULES, AND THEIR  
ASSOCIATIVITIES**

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# ABSTRACT OF THE DISSERTATION

Twisted intertwining operators, tensor products of twisted modules, and their  
associativities

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In this dissertation, we introduce a notion of twisted intertwining operator, which generalizes (untwisted) intertwining operators among (untwisted) modules for a vertex operator algebra  $V$  to the setting of twisted modules. This notion of twisted intertwining operator does not require that the associated automorphisms commute or have finite order. We establish the skew-symmetry and contragredient isomorphisms between spaces of twisted intertwining operators and also prove several properties of twisted intertwining operators. Using twisted intertwining operators, we introduce a notion of  $P(z)$ -tensor product  $W_1 \boxtimes_{P(z)} W_2$  of two objects  $W_1$  and  $W_2$  in a category of  $g$ -twisted  $V$ -modules for  $g$  in a subgroup  $G$  of  $\text{Aut}(V)$ . We give a set-theoretic construction of the  $P(z)$ -tensor product, where the contragredient module  $W_1 \boxtimes_{P(z)} W_2$  of  $W_1 \boxtimes_{P(z)} W_2$  can be realized as a subspace of  $(W_1 \otimes W_2)^*$ . We find a criterion for a linear functional  $\lambda \in (W_1 \otimes W_2)^*$  being contained in  $W_1 \boxtimes_{P(z)} W_2$ . Using the set-theoretic construction and the criterion, we prove that when the category  $\mathcal{C}$  of twisted modules satisfies suitable conditions, the associativity for twisted intertwining operators among objects in  $\mathcal{C}$  holds. As a result, we construct a family of associativity isomorphisms for the  $P(z)$ -tensor product bifunctors in a suitable sense.

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# Preface

Chapters 2-7 of this dissertation have appeared in the published paper [DH], which is a joint work with the author's advisor, Prof. Yi-Zhi Huang. Compare to [DH], the settings and notations in this dissertation are slightly different.

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# Chapter 1

## Introduction

This dissertation is on the tensor structure of the category of twisted modules for a vertex operator algebra.

The main results of this dissertation are a construction of  $P(z)$ -tensor product bifunctors, proving the associativity of twisted intertwining operators and a construction of the associativity isomorphisms for the  $P(z)$ -tensor product bifunctors.

### 1.1 The untwisted theory

#### 1.1.1 VOAs and their module categories I: formal calculus

Vertex operator algebras (VOAs) and vertex algebras (VAs) were primarily defined and studied by Frenkel, Lepowsky, and Meurman and by Borchers in their studies of infinite-dimensional Lie algebras and the Monster finite simple group (see [FLM4] and [B]). (In this dissertation, we would always work with VOAs rather than VAs). In quantum physics, VOAs arose naturally in the study of 2-dimensional conformal field theory (CFT) and string theory (see [MS] and [BPZ]). From an algebraic perspective, the notion of VOA can be viewed as a simultaneous generalization of the notion of Lie algebra and the notion of commutative associative algebra. The major axiom for

VOA is the *Jacobi identity*, that is, if  $(V, Y_V, \mathbf{1}, \omega)$  is a VOA, then for any  $u, v \in V$ ,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_V(u, x_1) Y_V(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_V(v, x_2) Y_V(u, x_1) \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_V(Y_V(u, x_0)v, x_2), \end{aligned} \quad (1.1)$$

where  $x_0, x_1, x_2$  are formal variables,  $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$  is a formal series, and  $Y_V(\cdot, x) \cdot$  is a linear map  $V \otimes V \rightarrow V[[x, x^{-1}]]$ , called the *vertex operator map*, satisfying the *lower truncation property*, i.e.,  $Y_V(u, x)v \in V((x))$  for any  $u, v \in V$  (see the full definition of a VOA in [LL]). the ‘three-variables’ delta functions in (1.1) should be understood as the following:

$$\begin{aligned} \delta \left( \frac{x_1 - x_2}{x_0} \right) &= \sum_{n \in \mathbb{Z}} x_0^{-n} (x_1 - x_2)^n = \sum_{n \in \mathbb{Z}, k \in \mathbb{N}} \binom{n}{k} (-1)^k x_0^{-n} x_1^{n-k} x_2^k, \\ \delta \left( \frac{-x_2 + x_1}{x_0} \right) &= \sum_{n \in \mathbb{Z}} x_0^{-n} (-x_2 + x_1)^n = \sum_{n \in \mathbb{Z}, k \in \mathbb{N}} \binom{n}{k} (-1)^{n-k} x_0^{-n} x_2^{n-k} x_1^k, \\ \delta \left( \frac{x_2 + x_0}{x_1} \right) &= \sum_{n \in \mathbb{Z}} x_1^{-n} (x_2 + x_0)^n = \sum_{n \in \mathbb{Z}, k \in \mathbb{N}} \binom{n}{k} x_1^{-n} x_2^{n-k} x_0^k. \end{aligned}$$

One should notice that  $\delta \left( \frac{x_1 - x_2}{x_0} \right)$  and  $\delta \left( \frac{-x_2 + x_1}{x_0} \right)$  are two different formal series.

As a generalization of both Lie algebras and commutative associative algebras, VOAs are expected to—and indeed do—admit a rich and deep representation theory. The major axiom for an *ordinary module*  $(W, Y_W)$  for a VOA  $V$  is also the *Jacobi identity for modules*, that is, for any  $u, v \in V$ ,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_W(Y_V(u, x_0)v, x_2), \end{aligned} \quad (1.2)$$

where  $W = \coprod_{n \in \mathbb{C}} W_{(n)}$  is a  $\mathbb{C}$ -graded vector space over  $\mathbb{C}$ , and  $Y_W : V \otimes W \rightarrow W[[x, x^{-1}]]$  is a linear map, called the *vertex operator map on module*, satisfying the

lower truncation property. (See the full definition of a module for a VOA in [LL]). For any  $n \in \mathbb{C}$ , the weight space  $W_{(n)}$  is the eigenspace of the Virasoro operator  $L(0)$  corresponding to the eigenvalue  $n$ . The notion of ordinary module was later generalized to the notion of *generalized module*, also known as *logarithmic module*, by Lepowsky, Huang, and Zhang (see [HLZ1]). For a generalized module  $W$  for a VOA, the Virasoro operator  $L(0)$  only need to be locally finite (i.e. the module can be written as direct sum of generalized eigenspaces of  $L(0)$ ), instead of semisimple. There is the gradation  $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ , where  $W_{[n]}$  is the generalized eigenspace of the Virasoro operator  $L(0)$  corresponding to the eigenvalue  $n$ . Since in this paper we would always work with generalized modules, for simplicity, we shall omit the word ‘generalized’ and say ‘module’ as short for generalized module.

It turns out that the category of modules for a VOA is extremely interesting. Roughly, a nice module category for a VOA has a sophisticated tensor category structure. We will mention some most important results on the category of modules for a VOA in the next section.

Now, it is time to mention the most crucial concept which gives birth to the giant theory on the tensor category structure for VOAs, **intertwining operators among three modules**. My favorite slogan of intertwining operator is the following one:

Q: How to define  $W_1 \boxtimes W_2$ ?

A: Define  $\text{Hom}_V(W_1 \boxtimes W_2, W_3)$  for every  $W_3$ , before we can define  $W_1 \boxtimes W_2$ .

This is such a universal principle in the spirit of Yoneda’s lemma. In fact, for any category  $\mathcal{C}$ , we know that  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathcal{C}}$ ,  $a \mapsto \text{Hom}_{\mathcal{C}}(a, -)$  is a fully faithful functor, where  $\mathbf{Set}^{\mathcal{C}}$  is the category of functors from  $\mathcal{C}$  to the category of sets. In other words, *knowing  $a$  is equivalent to knowing  $\text{Hom}_{\mathcal{C}}(a, -)$* . Here, one might be reminded of the saying “*The human essence is no abstraction inherent in each single individual. In its reality it is the ensemble of the social relations.*”

Indeed, intertwining operators bijectively correspond to elements in  $\text{Hom}_V(W_1 \boxtimes W_2, W_3)$ .

Given three  $V$ -modules  $W_i$ ,  $i = 1, 2, 3$ , an *intertwining operator* of type  $\binom{W_3}{W_1 W_2}$  is a linear map  $\mathcal{Y} : W_1 \otimes W_2 \rightarrow W_3\{x\}[\log x]$  satisfying certain axioms, among which the most important one is again, the *Jacobi identity for intertwining operators*, that is, for any  $u \in V$ ,  $w_1 \in W_1$ ,

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_{W_3}(u, x_1) \mathcal{Y}(w_1, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_1, x_2) Y_{W_2}(u, x_1) \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \mathcal{Y}(Y_{W_1}(u, x_0) w_1, x_2). \end{aligned} \quad (1.3)$$

Here, the codomain of  $\mathcal{Y}$  is

$$W_3\{x\}[\log x] = \left\{ \sum_{n \in \mathbb{C}} \sum_{k=0}^N w_{n,k} x^n (\log x)^k \mid N \in \mathbb{N}, w_{n,k} \in W_3 \right\}.$$

Notice that for any VOA  $V$  and  $V$ -module  $W$ , the vertex operators  $Y_V$  and  $Y_W$  are special intertwining operators of types  $\binom{V}{VV}$  and  $\binom{W}{VW}$ , respectively.

One thing we should mention here is that intertwining operator is weight-compatible in the following sense: for any intertwining operator  $\mathcal{Y}$  of type  $\binom{M}{W_1 W_2}$ , write

$$\mathcal{Y}(w_1, x) w_2 = \sum_{n \in \mathbb{C}} \sum_{k=0}^N \mathcal{Y}_{n,k}(w_1)(w_2) x^{-n-1} \log(x)^k, \quad (1.4)$$

where  $\mathcal{Y}_{n,k}(w_1) \in \text{Hom}_{\mathbb{C}}(W_2, W_3)$ , then

$$\begin{aligned} \text{wt}(\mathcal{Y}_{n,k}(w_1)(w_2)) &= \text{wt}(w_1) + \text{wt}(w_2) - n - 1, \\ &\text{for any homogeneous } w_1 \in W_1, w_2 \in W_2, \end{aligned} \quad (1.5)$$

where  $\text{wt}(w)$  is the  $L(0)$ -eigenvalue (i.e. *weight*) for an  $L(0)$ -generalized eigenvector  $w$ , and ‘homogeneous’ means being an  $L(0)$ -generalized eigenvector.

Jacobi identities for VOAs, modules for a VOA, and intertwining operators are very power tools for VOA theorists effectively developing many important theorems. The reason why they are so handy to use is the **formal calculus** theory, which was initiated by Frenkel, Lepowsky and Meurman in [FLM4] and further developed by Frenkel, Huang, and Lepowsky in [FHL]. In formal calculus, the properties of delta function were very well understood which allows people to do complicated calculation involving delta function conveniently without actually looking into every monomial in a formal series. Later, to fit in the setting of generalized module, the formal calculus theory was generalized to the **logarithmic formal calculus** theory by Huang, Lepowsky, and Zhang at the beginning of their series papers (see Chapter 3 in [HLZ2]).

The notion of a tensor product bifunctor  $\boxtimes$  on the module category for a VOA can be immediately written down once we have a suitable definition of intertwining operator. Given  $V$ -modules  $W_1$  and  $W_2$ , consider the following category:

Object:  $(M, \mathcal{Y})$ , where  $M$  is a  $V$ -module,

and  $\mathcal{Y}$  is an intertwining operator of type  $\begin{pmatrix} M \\ W_1 W_2 \end{pmatrix}$ ;

Morphism:  $\text{Hom}((M_1, \mathcal{Y}_1), (M_2, \mathcal{Y}_2)) = \{f \in \text{Hom}_V(M_1, M_2) \mid \mathcal{Y}_2(w, x) = f \circ \mathcal{Y}_1(w, x)\}$ .

Here, one should naturally extend  $f : M_1 \rightarrow M_2$  to a map  $f : M_1\{x\}[\log x] \rightarrow M_2\{x\}[\log x]$ . Then, we can have the following definition:

**Definition 1.1.1** ( $\odot$ ). *The tensor product  $(W_1 \boxtimes W_2, \mathcal{Y}_{\boxtimes})$  of  $W_1$  and  $W_2$  is the initial object in the above category.*

In other words, every intertwining operator of type  $\begin{pmatrix} M \\ W_1 W_2 \end{pmatrix}$  for some module  $M$  factors through the tensor product  $W_1 \boxtimes W_2$  in a unique way. This property uniquely characterizes  $W_1 \boxtimes W_2$ , up to an isomorphism in the above category. Moreover, it is easy to check that  $-\boxtimes-$  is a bifunctor.

It looks really nice. Actually, the module  $W_1 \boxtimes W_2$  is indeed isomorphic to the  $P(z)$ -tensor product we have constructed in this work.

However, if what we want is a tensor in the monoidal category sense, then not only the bifunctor itself, but an *associativity isomorphism* is needed as well. An associativity isomorphism is a natural isomorphism from the functor  $- \boxtimes (- \boxtimes -)$  to the functor  $(- \boxtimes -) \boxtimes -$  satisfying a certain *pentagon axiom*. If one tries to construct an associativity isomorphism for the tensor product  $\boxtimes$ , since the only tool one can use is the universal property of  $\boxtimes$ , one would realize that the most reasonable idea is to consider the following diagram:

$$\begin{array}{c}
 \begin{array}{ccc}
 & & W_1 \otimes \\
 & & (W_2 \boxtimes W_3 \{x_{23}\} [\log x_{23}]) \\
 & \nearrow & \downarrow \\
 (W_1 \boxtimes W_2 \{x_{12}\} [\log x_{12}]) & \longleftarrow W_1 \otimes W_2 \otimes W_3 & \longrightarrow W_1 \boxtimes (W_2 \boxtimes W_3) \\
 \otimes W_3 & \circ & \{x_{23}, x_{1,23}\} [\log x_{23}, \log x_{1,23}] \\
 & \searrow & \downarrow \\
 & & (W_1 \boxtimes W_2) \boxtimes W_3 \\
 & & \{x_{12}, x_{12,3}\} [\log x_{12}, \log x_{12,3}] \\
 & & \text{---}g\text{---} \rightarrow \text{the pushout?} \\
 & & \downarrow f \\
 & & \text{the pushout?}
 \end{array}
 \end{array}
 \tag{1.6}$$

where  $x_{12}, x_{12,3}, x_{23}, x_{1,23}$  are four formal variables. One could try the following strategy: (1) find the pushout; (2) prove that  $f$  and  $g$  are isomorphism; (3) use  $g^{-1} \circ f$  to produce a module map  $W_1 \boxtimes (W_2 \boxtimes W_3) \rightarrow (W_1 \boxtimes W_2) \boxtimes W_3$ .

The author personally has no idea about how to directly realize the above strategy. In fact knowing what is actually going on under the complex-analytic viewpoint, one can see that this strategy is far from feasible. The extremely deep complex geometry behind - the genus zero part of a CFT (see [H2]) - is perfectly hidden under this purely formal perspective.

This difficulty is the reason why one cannot go further without the complex-analytic viewpoint we would introduce in the next section. Without the complex-analytic approach, one can define the notions of VOAs, modules, intertwining operators, and therefore the tensor product bifunctors (Definition 1.1.1). One can also study individual modules, for example theories on construction of VOAs and modules. However, one cannot study the associativity of the tensor product.

Therefore, one need to step out of the scope of formal calculus, and begin to use a complex-analytic perspective, starting from replacing the formal variable  $x$  in an intertwining operator  $\mathcal{Y}(\cdot, x)$  by a complex variable  $z$ .

### 1.1.2 VOAs and their module categories II: complex analyticity

#### Revisiting: the notion of VOAs, modules, and intertwining operators

Continue from the previous subsection, we mentioned that one need to consider  $\mathcal{Y}(\cdot, z)$  where  $z$  is a complex variable. Let's start from the special cases, namely vertex operators  $Y_V$  for a VOA  $V$ , and  $Y_W$  for a  $V$ -module  $W$ . We first introduce some notations. For a  $\mathbb{C}$ -graded vector space  $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ , define

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}, \quad W' = \prod_{n \in \mathbb{C}} (W_{[n]})^*.$$

We know that  $W^* = \overline{W'}$ . If  $(W, Y_W)$  is a  $V$ -module, for any  $v \in V$  and  $w \in W$ , we write

$$Y_W(v, x)w = \sum_{n \in \mathbb{Z}} (Y_W)_n(v)(w)x^{-n-1}, \quad (1.7)$$

where  $(Y_W)_n(v) \in \text{End}(W)$  (this is the logarithm-free version of (1.4)). In particular this notation works for  $Y_V$  for a VOA  $V$ .

Let's go back to the notion of VOAs, modules, and intertwining operators. In fact, there is an equivalent complex-analytic version of definitions of VOAs, modules, and intertwining operators. For VOAs, one obtains an equivalent definition by replacing the Jacobi identity (1.1) by the following axiom: for any  $v_1, v_2, v_3 \in V$  and  $v' \in V'$ , there exists a rational function

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{(z_1 - z_2)^k z_1^l z_2^m}, \quad (1.8)$$

where  $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ , and  $k, l, m \in \mathbb{N}$ , such that the series

$$\begin{aligned} \langle v', Y_V(v_1, z_1)Y_V(v_2, z_2)v_3 \rangle &= \sum_{m, n \in \mathbb{Z}} \langle v', (Y_V)_m(v_1)(Y_V)_n(v_2)v_3 \rangle z_1^{-m-1} z_2^{-n-1}, \\ \langle v', Y_V(v_2, z_2)Y_V(v_1, z_1)v_3 \rangle &= \sum_{m, n \in \mathbb{Z}} \langle v', (Y_V)_m(v_2)(Y_V)_n(v_1)v_3 \rangle z_2^{-m-1} z_1^{-n-1}, \\ \langle v', Y_V(Y_V(v_1, z_1 - z_2)v_2, z_2)v_3 \rangle &= \sum_{m, n \in \mathbb{Z}} \langle v', (Y_V)_n((Y_V)_m(v_1)v_2)v_3 \rangle (z_1 - z_2)^{-m-1} z_2^{-n-1} \end{aligned}$$

are absolutely convergent to  $f(z_1, z_2)$ , on the region given by  $0 < |z_2| < |z_1|$ , the region given by  $0 < |z_1| < |z_2|$ , the region given by  $0 < |z_1 - z_2| < |z_2|$ , respectively.

This axiom is called the *duality property for VOAs*.

For modules for a VOA, one obtains an equivalent definition of modules for a VOA by replacing the Jacobi identity (1.2) by the following *duality property for modules*: for any  $v_1, v_2 \in V$ ,  $w \in W$  and  $w' \in W'$ , there exists a rational function

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{(z_1 - z_2)^k z_1^l z_2^m}, \quad (1.9)$$

where  $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ , and  $k, l, m \in \mathbb{N}$ , such that the series

$$\langle w', Y_W(v_1, z_1)Y_W(v_2, z_2)w \rangle = \sum_{m, n \in \mathbb{Z}} \langle w', (Y_W)_m(v_1)(Y_W)_n(v_2)w \rangle z_1^{-m-1} z_2^{-n-1},$$

$$\begin{aligned} \langle w', Y_W(v_2, z_2)Y_W(v_1, z_1)w \rangle &= \sum_{m, n \in \mathbb{Z}} \langle w', (Y_W)_n(v_2)(Y_W)_m(v_1)w \rangle z_2^{-n-1} z_1^{-m-1}, \\ \langle w', Y_W(Y_V(v_1, z_1 - z_2)v_2, z_2)w \rangle &= \sum_{m, n \in \mathbb{Z}} \langle w', (Y_W)_n((Y_V)_m(v_1)v_2)w \rangle (z_1 - z_2)^{-m-1} z_2^{-n-1} \end{aligned}$$

are absolutely convergent to  $f(z_1, z_2)$ , on the region given by  $0 < |z_2| < |z_1|$ , the region given by  $0 < |z_1| < |z_2|$ , the region given by  $0 < |z_1 - z_2| < |z_2|$ , respectively.

For  $p \in \mathbb{Z}$ , let  $l_p(z)$  be the single-valued branch of the multivalued logarithm function  $\text{Log}(z)$  on  $z \in \mathbb{C}^\times$  with  $\Im(l_p(z)) \in [2p\pi, 2(p+1)\pi)$  for any  $z \in \mathbb{C}^\times$ . By (1.5), for any  $z \in \mathbb{C}^\times$ ,  $p \in \mathbb{Z}$ , and intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1 W_2}$ , one can view  $\mathcal{Y}^p(w_1, z)w_2$  as an element in  $\overline{W_3}$ . Here,  $\mathcal{Y}^p(w_1, z)w_2$  is defined as the following:

$$\mathcal{Y}^p(w_1, z)w_2 = \sum_{n \in \mathbb{Z}, k \in \mathbb{N}} e^{(-n-1)l_p(z)} l_p(z)^k \mathcal{Y}_{n,k}(w_1)w_2 \in \overline{W_3}. \quad (1.10)$$

We denote  $\mathcal{Y}^0(w_1, z)w_2$  by  $\mathcal{Y}(w_1, z)w_2$ .

For intertwining operators, one obtains an equivalent definition of intertwining operators by replacing the Jacobi identity (1.3) by the following *duality property for modules*: for any  $v \in V$ ,  $w_1 \in W_1, w_2 \in W_2, w'_3 \in W_3$ , there exists a function

$$f(z_1, z_2) = \sum_{i=1}^N a_i (z_1 - z_2)^{k_i} z_1^{l_i} e^{m_i l_0(z_2)} l_0(z_2)^{p_i}, \quad (1.11)$$

where  $N \in \mathbb{Z}_+$ ,  $a_i \in \mathbb{C}$ ,  $k_i, l_i \in \mathbb{Z}$ ,  $m_i \in \mathbb{C}$ ,  $p_i \in \mathbb{N}$  for  $i = 1, \dots, N$ , such that the series

$$\begin{aligned} \langle w'_3, Y_{W_3}(v, z_1)\mathcal{Y}(w_1, z_2)w_2 \rangle &\stackrel{\text{def}}{=} \sum_{\substack{m \in \mathbb{Z}, \\ n \in \mathbb{C}, k \in \mathbb{N}}} \langle w'_3, (Y_{W_3})_m(v)\mathcal{Y}_{n,k}(w_1)w_2 \rangle \cdot \\ & z_1^{-m-1} e^{(-n-1)l_0(z_2)} l_0(z_2)^k, \quad (1.12) \end{aligned}$$

$$\langle w'_3, \mathcal{Y}(w_1, z_2)Y_{W_2}(v, z_1)w_2 \rangle \stackrel{\text{def}}{=} \sum_{\substack{m \in \mathbb{Z}, \\ n \in \mathbb{C}, k \in \mathbb{N}}} \langle w'_3, \mathcal{Y}_{n,k}(w_1)(Y_{W_2})_m(v)w_2 \rangle \cdot$$

$$\begin{aligned}
& z_1^{-m-1} e^{(-n-1)l_0(z_2)} l_0(z_2)^k, \\
\langle w'_3, \mathcal{Y}(Y_{W_1}(v, z_1 - z_2)w_1, z_2)w_2 \rangle & \stackrel{\text{def}}{=} \sum_{\substack{m \in \mathbb{Z}, \\ n \in \mathbb{C}, k \in \mathbb{N}}} \langle w'_3, (\mathcal{Y})_{n,k}((Y_{W_1})_m(v)w_1)w_2 \rangle \cdot \\
& (z_1 - z_2)^{-m-1} e^{(-n-1)l_0(z_2)} l_0(z_2)^k \quad (1.13)
\end{aligned}$$

are absolutely convergent to  $f(z_1, z_2)$ , on the region given by  $0 < |z_2| < |z_1|$ , the region given by  $0 < |z_1| < |z_2|$ , the region given by  $0 < |z_1 - z_2| < |z_2|$ , respectively.

Notice that in the duality properties for VOA, modules, and intertwining algebras, the *correlation function*  $f(z_1, z_2)$  is unique. Moreover, it depends on the four algebra/module/dual space elements multi-linearly, and depends on the complex variables  $z_1, z_2$  holomorphically.

### $P(z)$ -tensor products and their associativity isomorphisms

**Definition 1.1.2.** Let  $z \in \mathbb{C}^\times$ . We say  $I : W_1 \otimes W_2 \rightarrow \overline{W_3}$  is a  $P(z)$ -intertwining map of type  $\binom{W_3}{W_1 W_2}$ , if there exists an intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1 W_2}$ , such that  $I(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2$  (recall (1.10)).

**Definition 1.1.3.** Suppose  $W_1, W_2$  are  $V$ -modules for a VOA  $V$ , and  $z \in \mathbb{C}^\times$ . A  $P(z)$ -tensor product of  $W_1$  and  $W_2$  is a pair  $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ , where  $W_1 \boxtimes_{P(z)} W_2$  is a  $V$ -module, and  $\boxtimes_{P(z)}$  is a  $P(z)$ -intertwining map of type  $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$ , such that for any  $V$ -module  $M$  and  $P(z)$ -intertwining map  $I$  of type  $\binom{M}{W_1 W_2}$ , there uniquely exists a module map  $f : W_1 \boxtimes_{P(z)} W_2 \rightarrow M$ , such that  $\bar{f} \circ \boxtimes_{P(z)} = I$ , where  $\bar{f} : \overline{W_1 \boxtimes_{P(z)} W_2} \rightarrow \overline{W_3}$  is the map naturally induced by  $f$ .

We need to mention that Definition 1.1.3 cannot guarantee the existence of a  $P(z)$ -tensor product for two modules. However, under suitable conditions, we have a set-theoretic construction where we realize  $W_1 \boxtimes_{P(z)} W_2$  as a subquotient space of  $(W_1 \otimes W_2)^{**}$  (see Theorem 6.0.9). Throughout the background section, we assume that  $P(z)$ -tensor product for any two modules always exists.

For the  $P(z)$ -intertwining map  $\boxtimes_{P(z)}$ , we write  $\boxtimes_{P(z)}(w_1 \otimes w_2) = w_1 \boxtimes_{P(z)} w_2$ , called  $P(z)$ -*tensor product elements*, which is contained in  $\overline{W_1 \boxtimes_{P(z)} W_2}$ . It is not hard to prove that

$$\{\pi_n(w_1 \boxtimes_{P(z)} w_2) | n \in \mathbb{C}, w_1 \in W_1, w_2 \in W_2\} \quad (1.14)$$

spans  $W_1 \boxtimes_{P(z)} W_2$ , where  $\pi_n : \overline{W} \rightarrow W_{[n]}$  is the projection defined for any  $\mathbb{C}$ -graded vector space  $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ . By the definition of  $P(z)$ -intertwining map, there exists an intertwining operator, denoted by  $\mathcal{Y}_{\boxtimes_{P(z)}}$ , of type  $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$ , such that  $\mathcal{Y}_{\boxtimes_{P(z)}}(w_1, z)w_2 = w_1 \boxtimes_{P(z)} w_2$  for  $w_1 \in W_1, w_2 \in W_2$ .

A  $P(z)$ -tensor product is holomorphically dependent on a nonzero complex number  $z$ . One can think that  $\coprod_{z \in \mathbb{C}^\times} W_1 \boxtimes_{P(z)} W_2 \rightarrow \mathbb{C}^\times$  is an infinite-dimensional holomorphic vector bundle over  $\mathbb{C}^\times$ , whose local trivialization and transition map are given by the *parallel transport isomorphisms*, see (6.14). However, for  $w_1 \in W_1, w_2 \in W_2$ , the map  $z \mapsto w_1 \boxtimes_{P(z)} w_2$  is not a holomorphic section of the bundle, but can only be viewed as a **multivalued** analytic section.

Therefore, compare to Definition 1.1.1, only the definitions of  $P(z)$ -intertwining map and  $P(z)$ -tensor product allow one to study the theory using a complex-analytic point of view, which is necessary for constructing the associativity isomorphism.

We can do the similar thinking as diagram (1.6). The first problem is that the image of a  $P(z)$ -intertwining map is contained in the algebraic completion  $\overline{W}$  of a module  $W$ , instead of  $W$  itself. Therefore, in the following diagrams, the desired maps  $f$  and  $g$  cannot be easily defined.

$$\begin{array}{ccc} W_1 \otimes \overline{W_2 \boxtimes_{P(z_2)} W_3} & \longleftarrow & W_1 \otimes (W_2 \boxtimes_{P(z_2)} W_3) \\ \uparrow^{1_{W_1} \otimes (\cdot \boxtimes_{P(z_2)})} & \downarrow ? & \swarrow^{\cdot \boxtimes_{P(z_1)}} \\ W_1 \otimes W_2 \otimes W_3 & \xrightarrow{f} & \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \end{array} \quad (1.15)$$

$$\begin{array}{ccc}
& \overline{W_1 \boxtimes_{P(z_3)} W_2} \otimes W_3 & \longleftrightarrow & (W_1 \boxtimes_{P(z_3)} W_2) \otimes W_3 \\
& \nearrow^{(\cdot \boxtimes_{P(z_3)} \cdot) \otimes 1_{W_3}} & \downarrow ? & \nwarrow^{\cdot \boxtimes_{P(z_4)}} \\
W_1 \otimes W_2 \otimes W_3 & \xrightarrow{g} & \overline{(W_1 \boxtimes_{P(z_3)} W_2) \boxtimes_{P(z_4)} W_3} & 
\end{array} \tag{1.16}$$

In diagram (1.15), if the map ‘?’ wants to make the right triangle commutes, we know that ‘?’ has to send  $w_1 \otimes \pi_n(w_2 \boxtimes_{P(z_2)} w_3)$  to  $w_1 \boxtimes_{P(z_1)} \pi_n(w_2 \boxtimes_{P(z_2)} w_3)$  for any  $n \in \mathbb{C}$ , which hopefully means that it should send  $w_1 \otimes (w_2 \boxtimes_{P(z_2)} w_3)$  to  $\sum_{n \in \mathbb{C}} w_1 \boxtimes_{P(z_1)} (\pi_n(w_2 \boxtimes_{P(z_2)} w_3))$  (unless some pathological is happening here). Therefore, because the left triangle commutes, we know that

$$f(w_1 \otimes w_2 \otimes w_3) = \sum_{n \in \mathbb{C}} w_1 \boxtimes_{P(z_1)} (\pi_n(w_2 \boxtimes_{P(z_2)} w_3)). \tag{1.17}$$

The above equation is not rigorous, since the right-hand side, as an **infinite** sum of elements in  $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$ , is not well-defined. However, if for any

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))', w_1 \in W_1, w_2 \in W_2, w_3 \in W_3,$$

the series

$$\left\langle w', \mathcal{Y}_{\boxtimes_{P(z_1)}}(w_1, z_1) \mathcal{Y}_{\boxtimes_{P(z_2)}}(w_2, z_2) w_3 \right\rangle = \sum_{n \in \mathbb{C}} \left\langle w', \mathcal{Y}_{\boxtimes_{P(z_1)}}(w_1, z_1) \pi_n \left( \mathcal{Y}_{\boxtimes_{P(z_2)}}(w_2, z_2) w_3 \right) \right\rangle \tag{1.18}$$

is absolutely convergent, then  $\mathcal{Y}_{\boxtimes_{P(z_1)}}(w_1, z_1) \mathcal{Y}_{\boxtimes_{P(z_2)}}(w_2, z_2) w_3$  can be viewed as an element in  $\left( (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))' \right)^* = \overline{(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))''}$ . Throughout the background section, we assume that every module  $W$  that we consider satisfies that  $\dim W_{[n]} < \infty$ . In fact, this assumption occurs in the main results of this dissertation, for example Theorem 1.3.1. Under this assumption, we identify

$(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))''$  with  $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ . Therefore,

$$\mathcal{Y}_{\boxtimes_{P(z_1)}}(w_1, z_1) \mathcal{Y}_{\boxtimes_{P(z_2)}}(w_2, z_2) w_3 = w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)$$

can be viewed as an element in  $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$ , which rigorizes (1.17).

Therefore, if we assume that for any modules  $M_1$  and  $W_i$ ,  $i = 1, \dots, 4$ , that we consider, and any pair of intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of types  $\binom{W_4}{W_1 M_1}$  and  $\binom{M_1}{W_2 W_3}$ , the series

$$\langle w'_4, \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_2, z_2) w_3 \rangle = \sum_{n \in \mathbb{C}} \langle w'_4, \mathcal{Y}_1(w_1, z_1) \pi_n(\mathcal{Y}_2(w_2, z_2) w_3) \rangle, \quad (1.19)$$

is absolutely convergent, then we know that the function  $f$  in diagram (1.15) can be rigorously defined by (1.17).

However, it is not suitable to require that (1.19) is absolutely convergent for any  $z_1, z_2 \in \mathbb{C}^\times$ . By (1.12) in the duality of intertwining operator, we know that in the special case  $W_1 = V$ ,  $W_4 = M_1$ , and  $(\mathcal{Y}_1, \mathcal{Y}_2) = (Y_{M_1}, \mathcal{Y}_2)$ , (1.19) is absolutely convergent on and only on the region given by  $0 < |z_2| < |z_1|$ . We say that  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies *the convergence assumption for products* if (1.19) is absolutely convergent on the region given by  $0 < |z_2| < |z_1|$ .

Similarly, if we assume that for any modules  $M_2$  and  $W_i$ ,  $i = 1, \dots, 4$ , that we consider, and any pair of intertwining operators  $\mathcal{Y}_3, \mathcal{Y}_4$  of types  $\binom{W_4}{M_2 W_3}$  and  $\binom{M_2}{W_1 W_2}$ , the series

$$\langle w'_4, \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_3) w_2, z_4) w_3 \rangle = \sum_{n \in \mathbb{C}} \langle w'_4, \mathcal{Y}_3(\pi_n(\mathcal{Y}_4(w_1, z_3) w_2), z_4) w_3 \rangle, \quad (1.20)$$

is absolutely convergent, then we know that the function  $g$  in diagram (1.16) can be

rigorously defined by

$$g(w_1 \otimes w_2 \otimes w_3) = (w_1 \boxtimes_{P(z_3)} w_2) \boxtimes_{P(z_4)} w_3. \quad (1.21)$$

Also, by the special case  $W_1 = V$ ,  $M_2 = W_2$ , and  $(\mathcal{Y}_3, \mathcal{Y}_4) = (\mathcal{Y}_3, Y_{W_2})$ , (1.20) is absolutely convergent on and only on the region given by  $0 < |z_3| < |z_4|$ , because of (1.13) in the duality property for intertwining operators. We say that  $(\mathcal{Y}_3, \mathcal{Y}_4)$  satisfies *the convergence assumption for iterates* if (1.20) is absolutely convergent on the region given by  $0 < |z_3| < |z_4|$ .

Throughout the background section, we assume the for any intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$  of suitable types among modules that we consider,  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies the convergence assumption for products and  $(\mathcal{Y}_3, \mathcal{Y}_4)$  satisfies the convergence assumption for iterates. Under this assumption,  $f$  and  $g$  are always well-defined.

Once  $f$  and  $g$  in diagrams (1.15) and (1.16) are rigorously defined, one wants to find a module map  $A_{P(z_1), P(z_2)}^{P(z_3), P(z_4)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_3)} W_2) \boxtimes_{P(z_4)} W_3$  such that the following diagram commutes:

$$\begin{array}{ccc} W_1 \otimes W_2 \otimes W_3 & \xrightarrow{f} & \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \\ & & \swarrow \downarrow \\ & & \overline{(W_1 \boxtimes_{P(z_3)} W_2) \boxtimes_{P(z_4)} W_3} \\ & \nwarrow & \\ & & A_{P(z_1), P(z_2)}^{P(z_3), P(z_4)} \end{array}$$

By the definitions (1.17) and (1.21) of  $f$  and  $g$ , we know that

$$\overline{A_{P(z_1), P(z_2)}^{P(z_3), P(z_4)}} (w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_3)} w_2) \boxtimes_{P(z_4)} w_3. \quad (1.22)$$

Notice that if there exists a module isomorphism  $A_{P(z_1), P(z_2)}^{P(z_3), P(z_4)}$  satisfying (1.22), then it is unique and this map could be the associativity isomorphism we want.

Notice that  $f$  and  $g$  can be characterized by categorical properties. Given modules  $W_1, W_2, W_3$  and  $z_1, z_2 \in \mathbb{C}^\times$  with  $|z_2| < |z_1|$ , we define the following category, denoted

by  $\mathcal{C}_{\text{prod}, z_1, z_2}^{W_1, W_2, W_3}$ :

Object:  $(M, I)$ , where  $M$  is a  $V$ -module,

and  $I : W_1 \otimes W_2 \otimes W_3 \rightarrow \overline{M}$ , there exists intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$ ,

such that  $I(w_1 \otimes w_2 \otimes w_3) = \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3$ ,

Morphism:  $\text{Hom}((M_1, I_1), (M_2, I_2)) = \{f \in \text{Hom}_V(M_1, M_2) \mid I_2 = \overline{f} \circ I_1\}$ .

We claim that  $(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3), f)$  is an initial object of the category  $\mathcal{C}_{\text{prod}, z_1, z_2}^{W_1, W_2, W_3}$ . Similarly, for  $z_3, z_4 \in \mathbb{C}^\times$  with  $|z_3| < |z_4|$ , we can define a category  $\mathcal{C}_{\text{iter}, z_3, z_4}^{W_1, W_2, W_3}$ , such that  $((W_1 \boxtimes_{P(z_3)} W_2) \boxtimes_{P(z_4)} W_3, g)$  is an initial object of the category  $\mathcal{C}_{\text{iter}, z_3, z_4}^{W_1, W_2, W_3}$ .

It is not very hard to prove that the existence of the isomorphism  $A_{P(z_1), P(z_2)}^{P(z_3), P(z_4)}$  satisfying (1.22) is equivalent to the fact that the categories  $\mathcal{C}_{\text{prod}, z_1, z_2}^{W_1, W_2, W_3}$  and  $\mathcal{C}_{\text{iter}, z_3, z_4}^{W_1, W_2, W_3}$  are equal. Therefore, we should study the condition

$$\mathcal{C}_{\text{prod}, z_1, z_2}^{W_1, W_2, W_3} = \mathcal{C}_{\text{iter}, z_3, z_4}^{W_1, W_2, W_3}. \quad (1.23)$$

Explicitly, (1.23) means that for any intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of any suitable types, there exists intertwining operators  $\mathcal{Y}_3, \mathcal{Y}_4$  of suitable types, such that

$$\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3 = \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_3)w_2, z_4)w_3.$$

And vice versa, for any suitable  $\mathcal{Y}_3, \mathcal{Y}_4$ , there exist  $\mathcal{Y}_1, \mathcal{Y}_2$  such that the above equality holds.

In general, for arbitrary  $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$ , (1.23) looks impossible to prove. Inspiration still comes from the duality for intertwining operators, which says that

$$Y_{W_3}(u, z_1)\mathcal{Y}(w_1, z_2)w_2 = \mathcal{Y}(Y_{W_1}(u, z_1 - z_2)w_1, z_2)w_2, \quad (1.24)$$

for  $z_1, z_2 \in \mathbb{C}^\times$  with  $0 < |z_1 - z_2| < |z_2| < |z_1|$ . In other words, for any intertwining operators  $\mathcal{Y}$  of type  $\binom{W_3}{W_1 W_2}$ , we know that

$$\begin{aligned} (W_3, Y_{W_3}(\cdot, z_1)\mathcal{Y}(\cdot, z_2) \cdot) &= (W_3, \mathcal{Y}(Y_{W_1}(\cdot, z_1 - z_2)\cdot, z_2) \cdot) \\ &\in \text{Obj}\left(\mathcal{C}_{\text{prod}, z_1, z_2}^{W_1, W_2, W_3}\right) \cap \text{Obj}\left(\mathcal{C}_{\text{iter}, z_1 - z_2, z_2}^{W_1, W_2, W_3}\right). \end{aligned} \quad (1.25)$$

Therefore, maybe we should only expect that (1.23) holds if

$$z_3 = z_1 - z_2, \quad z_4 = z_2. \quad (1.26)$$

We say that the *associativity of intertwining operators* holds, if for any modules  $W_1, W_2, W_3$  under consideration, and any  $z_1, z_2 \in \mathbb{C}^\times$  with  $0 < |z_1 - z_2| < |z_2| < |z_1|$ ,

$$\mathcal{C}_{\text{prod}, z_1, z_2}^{W_1, W_2, W_3} = \mathcal{C}_{\text{iter}, z_1 - z_2, z_2}^{W_1, W_2, W_3}.$$

In the series of work [HL2]-[HL4], [H1] by Huang and Lepowsky, they proved the associativity of intertwining operator for the rational case (i.e. lower-bounded modules are all completely reducible) under certain assumptions. In the series of work [HLZ1]-[HLZ8], by Huang, Lepowsky and Zhang, they proved the associativity of intertwining operator for the general (i.e. non-rational) case under the same type of assumptions.

The major assumption they made is called the *convergence and extension assumption*. Roughly, it says that any product

$$\langle w'_4, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}(w_2, z_2)w_3 \rangle \quad (1.27)$$

among modules under consideration should be absolutely convergent on the region given by  $0 < |z_2| < |z_1|$ . Moreover, the function (1.27) converges to can be analytically

extended to the region given by  $0 < |z_1 - z_2| < |z_2|$ , on which the function can be written as an absolutely convergent series ‘of the form same as an iterate (of two intertwining operators), i.e., (1.28)’. Similarly, any iterate

$$\langle w'_4, \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle \quad (1.28)$$

among modules under consideration should be absolutely convergent on the region given by  $0 < |z_1 - z_2| < |z_2|$  to a function which can be analytically extended to the region given by  $0 < |z_2| < |z_1|$ , on which it can be written as an absolutely convergent series ‘of the form same as a product (of two intertwining operators), i.e., (1.27)’. (We would use product/iterate as short for product/iterate of two intertwining operator)

Here, a punchline could be:

If a product looks like an iterate, then it is equal to an iterate;  
conversely, if an iterate looks like a product, then it is equal to a product.

**Remark 1.1.4.** The convergence and extension assumption was later proved in [H4] for intertwining operators among  $C_1$ -cofinite modules, using regular singular differential equation theory, which enables Huang-Lepowsky-Zhang’s theory to be applied to many concrete examples. For example, for  $C_2$ -cofinite VOA, every grading-restricted module is  $C_1$ -cofinite. Also, recently, Huang proved that the  $P(z)$ -tensor product of two  $C_1$ -cofinite modules is still  $C_1$ -cofinite (see [H17]).

## 1.2 The twisted theory-a parallel generalization

Before we introduce the notion of twisted module, one thing need to be emphasized is that everything given in the sub-section 1.1.2, has been generalized to the twisted module case in this dissertation. The generalized notions are more complicated than their untwisted counterparts, mainly because the correlation functions are much more

complicated and multivalued. But the philosophy on the construction of the  $P(z)$ -tensor product and of the associativity isomorphism are the same.

The major difference is the following. Despite the involvedness of these complex-analytic assumptions (including convergence and extension assumptions), Huang-Lepowsky's and Huang-Lepowsky-Zhang's work mainly used the algebraic approach, i.e., the Jacobi identity and the formal calculus method. This is natural - one should always use the algebraic approach whenever "one can", i.e., when there is a Jacobi identity to use, because although often lengthy and extremely technical, formal delta-function calculus offers an effective way to do computations and prove theorems, and therefore has been foundational to the VOA theory.

However, in this dissertation, we have developed and mainly used a systematic complex-analytic approach. There are many good reasons to develop such an approach. One simple reason is that we do not know how to write down a Jacobi identity for the definition of the notion of a twisted intertwining operator for nonabelian orbifold theory (i.e., the group of automorphisms associated to the orbifold theory is nonabelian). Especially for the case when the associated group of automorphisms is both infinite and nonabelian, it seems very difficult to write down a Jacobi-identity version of definition of the twisted intertwining operator. Even if we have a Jacobi-identity version of definition, the duality property and therefore the complex-analytic approach are more natural and closer to the perspective of 2D CFTs in theoretical and mathematical physics.

### 1.2.1 Twisted modules for a VOA

The notion of (generalized) twisted modules for a VOA is a generalization of the notion of modules for a VOA. The theory on twisted modules corresponds to *orbifold CFTs*, which are CFTs constructed from known theories and their automorphisms. Therefore, the theory on twisted modules is also called 'orbifold theory'. The first ex-

ample of orbifold theory was the moonshine module VOA  $V^\natural$  constructed by Frenkel, Lepowsky and Meurman in mathematics (see [FLM2]-[FLM4]). Their construction of  $V^\natural$  proved the McKay-Thompson conjecture, profoundly relating number theory and finite group theory. Later, this VOA  $V^\natural$  played a major role in Borchers' proof of the rest of Conway–Norton Conjectures. FLM's construction introduced a new string theory, which was later interpreted by physicists as an “orbifold theory”. In string theory, the more general systematic study of orbifold CFTs was started by Dixon, Harvey, Vafa and Witten [DHVW1] [DHVW2]. See [H15] for an exposition of general results, conjectures and open problems in the construction of orbifold CFTs using the approach of the representation theory of vertex operator algebras.

Let  $V$  be a VOA, and  $g \in \text{Aut}(V)$ . A  $g$ -twisted  $V$ -module is a pair  $(W, Y_W)$ , where  $W = \coprod_{n \in \mathbb{C}} W_{[n]}$  is a  $\mathbb{C}$ -graded vector space, and  $Y_W : V \otimes W \rightarrow W\{x\}[\log x]$  is a linear map, satisfying certain axioms. The most important two axioms are the duality property and the equivariance property. The duality property for a  $g$ -twisted  $V$ -module is the same as the duality property for a (untwisted)  $V$ -module, while the only difference is that the correlation function is no longer a rational function in  $z_1, z_2$ , but a function of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^N a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t} \quad (1.29)$$

for  $N \in \mathbb{N}$ ,  $m_1, \dots, m_N, n_1, \dots, n_N \in \mathbb{C}$  and  $t \in \mathbb{N}$ . The form (1.29) is already much more complicated than the form of correlation function (1.11) in the duality of (untwisted) intertwining operators. It looks wild. Fortunately, we have a control on its multivaluedness, which is called the *equivariance property*:

$$(Y_W)^{p+1}(gv, x) = (Y_W)^p(v, x), \text{ for any } p \in \mathbb{Z}, v \in V. \quad (1.30)$$

(see (1.10) for the definition of  $(Y_W)^{p+1}$ )

It is natural to expect that Huang-Lepowsky-Zhang's result ([HLZ1]-[HLZ8]) has generalizations in orbifold theory.

In [K3], Kirillov Jr. stated that the category of  $g$ -twisted modules for a vertex operator algebra  $V$  for all  $g$  in a finite subgroup  $G$  of the automorphism group of  $V$  is a  $G$ -equivariant fusion category ( $G$ -crossed braided (tensor) category in the sense of Turaev [Tu2]). For general  $V$ , this is certainly not true. The vertex operator algebra  $V$  must satisfy certain conditions. Here is a precise conjecture formulated by Huang in [H10]:

**Conjecture 1.2.1.** *Let  $V$  be a vertex operator satisfying the following three conditions*

1. *For  $n < 0$ ,  $V_{(n)} = 0$ ;  $V_{(0)} = \mathbb{C}\mathbf{1}$ ; and as a  $V$ -module,  $V$  is equivalent to its contragredient  $V$ -module  $V'$ .*
2. *Every lower-bounded generalized  $V$ -module is completely reducible.*
3.  *$V$  is  $C_2$ -cofinite.*

*Let  $G$  be a finite group of automorphisms of  $V$ . Then the category of  $g$ -twisted  $V$ -modules for all  $g \in G$  is a  $G$ -crossed braided tensor category.*

We also conjecture that the category of  $g$ -twisted  $V$ -modules for all  $g \in G$  is a  $G$ -crossed modular tensor category in a suitable sense. Since the definitions of  $G$ -crossed modular tensor category in [K3] and [Tu2] are different, more work needs to be done to find out which definition is the correct one for the category of twisted modules for a vertex operator algebra. But we do believe that this stronger  $G$ -crossed modular tensor category conjecture should be true in a suitable sense.

This dissertation has constructed a tensor product bifunctor, and proven the existence of an associativity isomorphism under certain assumptions, among which the most crucial one is called the *convergence and extension assumption*. See section 1.3 for details. These results are all necessary for proving Conjecture 1.2.1.

In the case that  $G$  is trivial, i.e. the untwisted theory, Conjecture 1.2.1 and even the stronger  $G$ -crossed modular tensor category conjecture is true by Huang-Lepowsky-Zhang [HLZ1]-[HLZ8] and Huang [H4]-[H7]. Thus, the  $G$ -crossed modular tensor category conjecture is a natural generalization of its untwisted version.

In the case that the fixed point subalgebra  $V^G$  of  $V$  under  $G$  satisfies the conditions in Conjecture 1.2.1 above, the category of  $V^G$ -modules is a modular tensor category. In this case, Conjecture 1.2.1 can be proved using the modular tensor category structure on the category of  $V^G$ -modules and the results on tensor categories by Kirillov Jr. [K1] [K2] [K3] and Müger [Mü1] [Mü2]. In the special case that  $G$  is a finite cyclic group and  $V$  satisfies the conditions in Conjecture 1.2.1, Carnahan-Miyamoto [CM] proved that  $V^G$  also satisfies the conditions in Conjecture 1.2.1. In the case that  $G$  is a finite cyclic group and  $V$  is in addition a holomorphic vertex operator algebra (meaning that the only irreducible  $V$ -module is  $V$  itself), Conjecture 1.2.1 can be obtained as a consequence of the results of van Ekeren-Möller-Scheithauer [EMS] and Möller [Mö] on the modular tensor category of  $V^G$ -modules. Assuming that  $G$  is a finite group containing the parity involution and that the category of grading-restricted  $V^G$ -modules has a natural structure of vertex tensor category structure in the sense of [HL1], McRae [Mc] constructed a nonsemisimple  $G$ -crossed braided tensor category structure on the category of grading-restricted (generalized)  $g$ -twisted  $V$ -modules.

For a general finite group  $G$ , the conjecture that the fixed point subalgebra  $V^G$  of  $V$  under  $G$  also satisfies the conditions in Conjecture 1.2.1 is still open and seems to be a difficult problem. On the other hand, using twisted modules and twisted intertwining operators to construct  $G$ -crossed braided tensor categories seems to be a more conceptual and direct approach. If this approach works, we expect that the category of  $V^G$ -modules can also be studied using the  $G$ -crossed braided tensor category structure on the category of twisted  $V$ -modules.

In the case that the vertex operator algebra  $V$  does not satisfy the three conditions

in Conjecture 1.2.1 and/or the group  $G$  is not finite, it is not even clear what the precise conjecture should be. This was proposed as an open problem in [H10].

## 1.2.2 Twisted intertwining operators

Intertwining operators among twisted modules (what we called *twisted intertwining operators*) associated to commuting automorphisms of finite order appeared implicitly in the work [FFR] of Feingold, Frenkel and Ries and were introduced explicitly by Xu in [X] in terms of a generalization of the Jacobi identity for twisted modules. Xu's Jacobi identity works because in [X], only modules twisted by automorphisms in a *finite abelian group* are considered.

In [H9], Huang introduced a definition of twisted intertwining operators among modules twisted by noncommuting automorphisms. In that definition, the correlation function in the duality property satisfies a special explicit form. It turns out that the definition in [H9] is not general enough to study orbifold theory associated to a nonabelian group of automorphisms, since the explicit form of the correlation function implicitly indicates the commutativity of the automorphisms involved.

In this dissertation, we have introduced the most general notion of twisted intertwining operator, where no explicit form needs to be satisfied. In this definition, the correlation functions are multivalued functions whose single-valued branches are indexed by elements of the fundamental group of some configuration space, and are determined only by the image of an anti-homomorphism from the fundamental group to the group of automorphisms  $\langle g_1, g_2 \rangle$  generated by  $g_1, g_2$  (if the twisted intertwining operator is of type  $\begin{pmatrix} g_1 g_2 \\ g_1 \ g_2 \end{pmatrix}$ ). This definition of twisted intertwining operator is general enough for studying the orbifold theory associated to a nonabelian group of automorphisms. In order to give the correct notion of  $P(z)$ -tensor product of twisted modules, we need to use the most general twisted intertwining operators. If we use only a certain special set of twisted intertwining operators as in [H9] to define and construct the

$P(z)$ -tensor product bifunctor, we would obtain a quotient of the correct  $P(z)$ -tensor product structure. The notion of  $P(z)$ -tensor product has been foundational for the theory from the beginning. See the full definition of twisted intertwining operator in Definition 3.0.23.

Moreover, based on our definition of twisted intertwining operators, we have proved some properties of twisted intertwining operators that are essential for the construction of  $G$ -braided vertex tensor categories. For example, we have constructed skew-symmetry isomorphisms  $\Omega_{\pm}$  and contragredient isomorphisms  $A_{\pm}$  between spaces of twisted intertwining operators. Although these isomorphisms occurred and played crucial roles in the untwisted theory, since we have used our most general notion of intertwining operator, the proof of their existence was new and used the complex-analytic approach.

### 1.2.3 Associativity of twisted intertwining operators

The strategy of proving the associativity of twisted intertwining operators is the same as Huang-Lepowsky's and Huang-Lepowsky-Zhang's proof of the associativity of intertwining operators among untwisted modules (see the subsection 1.1.2.2).

The difference is that in their proof, the most crucial tool developed is a condition called  $P(z)$ -compatibility. Since the untwisted intertwining operator can be defined using the Jacobi identity, which is algebraic. The  $P(z)$ -compatibility condition in [HLZ4] is a purely algebraic statement, which is invalid under our notion of twisted intertwining operator and the complex analytic setting. To solve this problem, we have introduced a new condition, which is a complex-analytic statement although we also call it the  $P(z)$ -compatibility condition. It looks very different from the algebraic version of  $P(z)$ -compatibility condition in [HLZ4]. Whether they are equivalent when the twisted modules considered are actually untwisted is still unclear, which is an interesting unsolved problem.

The complex analytic  $P(z)$ -compatibility condition serves the same function as the algebraic one, in the sense that we still can develop the same useful tool in our complex analytic setting (See [DH]). Again, since our notions of twisted intertwining operator and of  $P(z)$ -compatibility condition are very different, the method of developing these tools is new.

Also, the convergence and extension assumptions in the dissertation are stronger than their untwisted version in [H1] and [HLZ7]. This is because of two reasons. First, the multivaluedness occurring in twisted modules and twisted intertwining operators is much more complicated than the untwisted case. Therefore, some result can be easily proved in the untwisted case is no longer true in the twisted setting. Therefore, we indeed need stronger assumptions. Second, from the experience of Huang's work on proving these assumptions using regular singular differential equation theory (see [H4]), although stronger, the assumptions we have made are still reasonable and hopeful to be proved under certain cofiniteness condition using the twisted generalization of Huang's work [H4], which is another ongoing project (see [Ta1] and [Ta2]).

#### 1.2.4 $G$ -crossed braided tensor category

The notion of  $G$ -crossed braided tensor category was firstly introduced by Turaev in [Tu1] and [Tu2], where this notion was designed to study certain TQFTs with an extra symmetry described by a group  $G$ .

Let  $G$  be a group.

**Definition 1.2.2.** *A  $G$ -equivariant category is a tuple*

$$\left(\mathcal{C}, \{\phi_g\}_{g \in G}, \{\alpha_{g,h}\}_{g,h \in G}\right),$$

where

1.  $\mathcal{C}$  is an abelian category over  $\mathbb{C}$  with a  $G$ -grading, namely, a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g, \quad (1.31)$$

where each  $\mathcal{C}_g$  is an abelian full subcategory of  $\mathcal{C}$ , and  $\text{Hom}_{\mathcal{C}}(\mathcal{C}_g, \mathcal{C}_h) = 0$  if  $g \neq h$ ;

2.  $\{\phi_g\}_{g \in G}$  is a family of endofunctors  $\phi_g : \mathcal{C} \rightarrow \mathcal{C}$ , and  $\{\alpha_{g,h}\}_{g,h \in G}$  is a family of natural isomorphisms  $\alpha_{g,h} : \phi_g \circ \phi_h \Rightarrow \phi_{gh}$ .

Moreover, the following conditions hold:

1.  $\phi_g(W) \in \text{Object}(\mathcal{C}_{ghg^{-1}})$ , for any  $g, h \in G$ , and  $W \in \text{Object}(\mathcal{C}_h)$ ;
2.  $\phi_g = \text{id}_{\mathcal{C}}$ ;
3. the following diagram commutes:

$$\begin{array}{ccc} \phi_f \circ (\phi_g \circ \phi_h) = (\phi_f \circ \phi_g) \circ \phi_h & \xrightarrow{(\alpha_{f,g})_{\phi_h(-)}} & \phi_{fg} \circ \phi_h \\ \phi_f((\alpha_{g,h})_-) \downarrow & & \downarrow \alpha_{fg,h} \\ \phi_f \circ \phi_{gh} & \xrightarrow{\alpha_{f,gh}} & \phi_{fgh} \end{array}$$

**Definition 1.2.3.** A  $G$ -crossed braided tensor category is a tuple

$$\left( \mathcal{C}, \{\phi_g\}_{g \in G}, \{\alpha_{g,h}\}_{g,h \in G}, V, - \otimes -, a_{-, -, -}, l_-, r_-, \{J_{-, -}^g\}_{g \in G}, B_{-, -} \right),$$

where

1.  $(\mathcal{C}, \{\phi_g\}_{g \in G}, \{\alpha_{g,h}\}_{g,h \in G})$  is a  $G$ -equivariant category;
2.  $(\mathcal{C}, V, - \otimes -, a_{-, -, -}, l_-, r_-)$  is an abelian monoidal category over  $\mathbb{C}$ , where  $V$  is the unit object,  $- \otimes -$  is the tensor product bifunctor,  $a_{-, -, -}$  is the associativity isomorphism, and  $l_-, r_-$  are the left and right unital isomorphisms, respectively. We require that the bifunctor  $- \otimes -$  is  $\mathbb{C}$ -bilinear;

3.  $W_1 \otimes W_2 \in \text{Object}(\mathcal{C}_{gh})$ , for any  $W_1 \in \text{Object}(\mathcal{C}_g)$  and  $W_2 \in \text{Object}(\mathcal{C}_h)$ ;
4. For each  $g \in G$ ,  $J^g : \phi_g \circ (- \otimes -) \Rightarrow \phi_g(-) \otimes \phi_g(-)$  is a natural isomorphism. Moreover,  $(\phi_g, J^g)$  forms a tensor functor. Namely,  $V \cong \phi_g(V)$  and the follow diagram commutes:

$$\begin{array}{ccc}
(\phi_g(W_1) \otimes \phi_g(W_2)) \otimes \phi_g(W_3) & \xrightarrow{a_{\phi_g(W_1), \phi_g(W_2), \phi_g(W_3)}} & \phi_g(W_1) \otimes (\phi_g(W_2) \otimes \phi_g(W_3)) \\
\downarrow J_{W_1, W_2}^g \otimes \text{id}_{\phi_g(W_3)} & & \text{id}_{\phi_g(W_1)} \otimes J_{W_2, W_3}^g \downarrow \\
\phi_g(W_1 \otimes W_2) \otimes \phi_g(W_3) & & \phi_g(W_1) \otimes (\phi_g(W_2 \otimes W_3)) \\
\downarrow J_{W_1 \otimes W_2, W_3}^g & & J_{W_1, W_2 \otimes W_3}^g \downarrow \\
\phi_g((W_1 \otimes W_2) \otimes W_3) & \xrightarrow{\phi_g(a_{W_1, W_2, W_3})} & \phi_g(W_1 \otimes (W_2 \otimes W_3))
\end{array}$$

5.  $B_{W_1, W_2} : W_1 \otimes W_2 \rightarrow \phi_g(W_2) \otimes W_1$  is a natural transformation, where  $W_1 \in \text{Object}(\mathcal{C}_g)$ ,  $W_2 \in \text{Object}(\mathcal{C})$ .

Moreover, the following diagrams commute:

1. For  $g_1, g_2 \in G$ ,  $W_1 \in \text{Object}(\mathcal{C}_{g_1})$ ,  $W_2 \in \text{Object}(\mathcal{C}_{g_2})$ , and  $W_3 \in \text{Object}(\mathcal{C})$ ,

$$\begin{array}{ccc}
(W_1 \otimes W_2) \otimes W_3 & \xrightarrow{a_{W_1, W_2, W_3}} & W_1 \otimes (W_2 \otimes W_3) \\
B_{W_1, W_2} \otimes \text{id}_{W_3} \downarrow & & \downarrow B_{W_1, W_2 \otimes W_3} \\
(\phi_{g_1}(W_2) \otimes W_1) \otimes W_3 & & \phi_{g_1}(W_2 \otimes W_3) \otimes W_1 \\
a_{\phi_{g_1}(W_2), W_1, W_3} \downarrow & & \downarrow (J_{W_2, W_3}^{g_1})^{-1} \otimes \text{id}_{W_1} \\
\phi_{g_1}(W_2) \otimes (W_1 \otimes W_3) & & (\phi_{g_1}(W_2) \otimes \phi_{g_1}(W_3)) \otimes W_1 \\
& \searrow \text{id}_{\phi_{g_1}(W_2)} \otimes B_{W_1, W_3} & \downarrow a_{\phi_{g_1}(W_2), \phi_{g_1}(W_3), W_1} \\
& & \phi_{g_1}(W_2) \otimes (\phi_{g_1}(W_3) \otimes W_1)
\end{array}$$

and

$$\begin{array}{ccc}
W_1 \otimes (W_2 \otimes W_3) & \xrightarrow{(a_{W_1, W_2, W_3})^{-1}} & (W_1 \otimes W_2) \otimes W_3 \\
\text{id}_{W_1} \otimes B_{W_2, W_3} \downarrow & & \downarrow B_{W_1 \otimes W_2, W_3} \\
W_1 \otimes (\phi_{g_2}(W_3) \otimes W_2) & & \phi_{g_1 g_2}(W_3) \otimes (W_1 \otimes W_2) \\
(a_{W_1, \phi_{g_2}(W_3), W_2})^{-1} \downarrow & & \downarrow (a_{\phi_{g_1 g_2}(W_3), W_1, W_2})^{-1} \\
(W_1 \otimes \phi_{g_2}(W_3)) \otimes W_2 & & (\phi_{g_1 g_2}(W_3) \otimes W_1) \otimes W_2 \\
& \searrow B_{W_1, \phi_{g_2}(W_3)} \otimes \text{id}_{W_2} & \downarrow ((\alpha_{g_1, g_2})_{W_3}^{-1} \otimes \text{id}_{W_1}) \otimes \text{id}_{W_2} \\
& & (\phi_{g_1}(\phi_{g_2}(W_3)) \otimes W_1) \otimes W_2
\end{array}$$

2. For  $g, h \in G$ , and  $W_1, W_2 \in \text{Object}(\mathcal{C})$ ,

$$\begin{array}{ccc}
\phi_g(\phi_h(W_1)) \otimes \phi_g(\phi_h(W_2)) & \xrightarrow{(\alpha_{g,h})_{W_1} \otimes (\alpha_{g,h})_{W_2}} & \phi_{gh}(W_1) \otimes \phi_{gh}(W_2) \\
\downarrow J_{\phi_h(W_1), \phi_h(W_2)}^g & & \downarrow J_{W_1, W_2}^{gh} \\
\phi_g(\phi_h(W_1) \otimes \phi_h(W_2)) & & \phi_{gh}(W_1 \otimes W_2) \\
\downarrow \phi_g(J_{W_1, W_2}^h) & & \downarrow (\alpha_{g,h})_{W_1 \otimes W_2} \\
\phi_g(\phi_h(W_1 \otimes W_2)) & \xrightarrow{(\alpha_{g,h})_{W_1 \otimes W_2}} & \phi_{gh}(W_1 \otimes W_2)
\end{array}$$

3. For  $g, g_1 \in G$ ,  $W_1 \in \text{Object}(\mathcal{C}_{g_1})$ , and  $W_2 \in \text{Object}(\mathcal{C})$ ,

$$\begin{array}{ccc}
\phi_g(W_1) \otimes \phi_g(W_2) & \xrightarrow{J_{W_1, W_2}^g} & \phi_g(W_1 \otimes W_2) \\
B_{\phi_g(W_1), \phi_g(W_2)} \downarrow & & \downarrow \phi_g(B_{W_1, W_2}) \\
\phi_{gg_1 g^{-1}}(\phi_g(W_2)) \otimes \phi_g(W_1) & & \phi_g(\phi_{g_1}(W_2) \otimes W_1) \\
(\alpha_{gg_1 g^{-1}, g})_{W_2} \otimes \text{id}_{\phi_g(W_1)} \downarrow & & \downarrow (J_{\phi_{g_1}(W_2), W_1}^g)^{-1} \\
\phi_{gg_1}(W_2) \otimes \phi_g(W_1) & \xrightarrow{(\alpha_{g, g_1})_{W_2}^{-1} \otimes \text{id}_{\phi_g(W_1)}} & \phi_g(\phi_{g_1}(W_2)) \otimes \phi_g(W_1)
\end{array}$$

### 1.3 Main results

Let  $(V, Y, \omega, \mathbf{1})$  be a vertex operator algebra. Let  $\mathcal{C}$  be a category of twisted  $V$ -modules, namely, a full subcategory of the category of all  $g$ -twisted  $V$ -modules, where

$g \in \text{Aut}(V)$ . We say that the category  $\mathcal{C}$  satisfies the **associativity of twisted intertwining operators**, if the following two properties are satisfied.

1. Let  $W_1, W_2, W_3, M_1, W_4$  be any  $g_1^-$ ,  $g_2^-$ ,  $g_3^-$ ,  $g_2g_3^-$ ,  $g_1g_2g_3^-$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. For any twisted intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of types  $\binom{W_4}{W_1 M_1}$ ,  $\binom{M_1}{W_2 W_3}$ , respectively, there exists a  $g_1g_2$ -twisted  $V$ -module  $M_2$  in  $\mathcal{C}$ , and twisted intertwining operators  $\mathcal{Y}_3, \mathcal{Y}_4$  of type  $\binom{M_2}{W_1 W_2}$ ,  $\binom{W_4}{M_2 W_3}$ , respectively, such that for any  $w_i \in W_i$ ,  $i = 1, 2, 3$ , and  $w'_4 \in W'_4$ , the equation (1.33) holds on the region given by

$$\begin{aligned} 0 < |z_1 - z_2| < |z_2| < |z_1|, \\ |\arg(z_1 - z_2) - \arg(z_1)| < \frac{\pi}{2}, |\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}. \end{aligned} \quad (1.32)$$

2. Let  $W_1, W_2, W_3, M_2, W_4$  be any  $g_1^-$ ,  $g_2^-$ ,  $g_3^-$ ,  $g_1g_2^-$ ,  $g_1g_2g_3^-$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. For any twisted intertwining operators  $\mathcal{Y}_3, \mathcal{Y}_4$  of types  $\binom{M_2}{W_1 W_2}$ ,  $\binom{W_4}{M_2 W_3}$ , respectively, there exists a  $g_2g_3$ -twisted  $V$ -module  $M_1$  in  $\mathcal{C}$ , and twisted intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of type  $\binom{W_4}{W_1 M_1}$ ,  $\binom{M_1}{W_2 W_3}$ , respectively, such that for any  $w_i \in W_i$ ,  $i = 1, 2, 3$ , and  $w'_4 \in W'_4$ , the equation (1.33) holds on the region given by (1.32).

$$\langle w'_4, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3 \rangle = \langle w'_4, \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle. \quad (1.33)$$

We have the following theorem.

**Theorem 1.3.1.** *Suppose  $\mathcal{C}$  satisfies assumptions 6.0.7, 8.0.1, and 8.0.5. In other words,*

1. *For any twisted module  $W$  in  $\mathcal{C}$ , the nilpotent part  $L_W(0)_N$  of  $L_W(0)$  is nilpotent on  $W$ .*

2. For any twisted module  $W$  in  $\mathcal{C}$ ,  $L_W(1)$  is locally nilpotent.
3. For any  $z \in \mathbb{C}^\times$ , and twisted modules  $W_1$  and  $W_2$  in  $\mathcal{C}$ ,  $W_1 \square_{P(z)} W_2$  is also in  $\mathcal{C}$
4. The contragredient of an object in  $\mathcal{C}$  is also in  $\mathcal{C}$ . The direct sum of two objects in  $\mathcal{C}$  is also in  $\mathcal{C}$ .
5. For any twisted module  $W$  in  $\mathcal{C}$  and  $n \in \mathbb{C}$ ,  $\dim(W_{[n]}) < \infty$ . Equivalently,  $W \cong W''$  for any twisted module  $W$  in  $\mathcal{C}$ .
6. Let  $W_1, W_2, W_3, M_1, M_2, W_4$  be any  $g_1$ -,  $g_2$ -,  $g_3$ -,  $g_2g_3$ -,  $g_1g_2$ -,  $g_1g_2g_3$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. Let  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$  be any twisted intertwining operators of types  $\binom{W_4}{W_1 M_1}$ ,  $\binom{M_1}{W_2 W_3}$ ,  $\binom{M_2}{W_1 W_2}$ ,  $\binom{W_4}{M_2 W_3}$ , respectively.
  - (a)  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies the regular singular point property and the convergence and extension property for products.
  - (b)  $(\mathcal{Y}_3, \mathcal{Y}_4)$  satisfies the regular singular point property and the convergence and extension property for iterates.
7.  $\mathcal{C}$  is closed under taking image in the sense of Definition 7.0.22.

Furthermore, we assume

8. Let  $W_1, W_2, W_3, M_1, \mathcal{Y}_1, \mathcal{Y}_2$  be as in 6. For any  $w_3 \in W_3, w'_4 \in W'_4, z_1, z_2 \in \mathbb{C}$  satisfying (1.32), and  $n \in \mathbb{C}$ , the functional  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) \in (W_1 \otimes W_2)^*$  defined in (9.4) is  $P(z_1 - z_2)$ - $\mathcal{C}$ -embeddable.

Then, the associativity of twisted intertwining operators for  $\mathcal{C}$  holds.

The  $\mathcal{C}$ -embeddability mentioned in condition 8. in Theorem 1.3.1 can be easily proved under some conditions. Therefore, we have the following two Corollaries.

**Corollary 1.3.2.** *Suppose  $\mathcal{C}$  satisfies Condition 1-7 in Theorem 1.3.1. Furthermore, we assume that for any  $g \in \{h \in \text{Aut}(V) \mid \text{Obj}(\mathcal{C}_h) \neq \emptyset\}$ , any  $g$ -twisted module generated by one element is an object of  $\mathcal{C}$ . Then, the associativity of twisted intertwining operators for  $\mathcal{C}$  holds.*

**Corollary 1.3.3.** *Suppose  $\mathcal{C}$  satisfies Condition 1-7 in Theorem 1.3.1. Furthermore, we assume*

1. *For any  $g \in \{h \in \text{Aut}(V) \mid \text{Obj}(\mathcal{C}_h) \neq \emptyset\}$ , any grading-restricted  $g$ -twisted module generated by one element is an object of  $\mathcal{C}$ .*
2. *Let  $W_1, W_2, W_3, M_1, \mathcal{Y}_1, \mathcal{Y}_2$  be as in condition 7. in Theorem 1.3.1. There exists  $N_{\mathcal{Y}_1, \mathcal{Y}_2} \in \mathbb{N}$  such that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) = 0$ , for any  $n \in \mathbb{C}$  with  $\Re(n) < -N_{\mathcal{Y}_1, \mathcal{Y}_2}$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , and  $(z_1, z_2)$  satisfying (11.1).  
(See (9.4) for the definition of  $\tilde{\lambda}_n^{(2)}$ .)*

*Then, the associativity of twisted intertwining operators for  $\mathcal{C}$  holds.*

Suppose that  $G$  is a subgroup of  $\text{Aut}(V)$ . We denote by  $\mathcal{GM}_{gr}(G)$  the category of all grading-restricted  $g$ -twisted generalized  $V$ -modules for all  $g \in G$ . As a corollary of 1.3.1, we have

**Corollary 1.3.4.** *Let  $G$  be a subgroup of  $\text{Aut}(V)$ . Assume that the following conditions are satisfied:*

1. *For  $g \in G$ , there are only finitely many irreducible grading-restricted  $g$ -twisted  $V$ -modules.*
2. *For  $g \in G$ , every grading-restricted  $g$ -twisted  $V$ -module is completely reducible.*
3. *For  $g_1, g_2 \in G$ , and grading-restricted  $g_1$ -,  $g_2$ -,  $g_1 g_2$ -twisted  $V$ -modules  $W_1, W_2$ , and  $W_3$ , the fusion rule  $N_{W_1 W_2}^{W_3} := \dim \mathcal{V}_{W_1 W_2}^{W_3}$  is finite.*

4. For any  $g \in G$ , and irreducible grading-restricted  $g$ -twisted  $V$ -module  $W$ ,  $L_W(0)_N$  is nilpotent.
5. Let  $W_1, W_2, W_3, M_1, M_2, W_4$  be any  $g_1^-$ ,  $g_2^-$ ,  $g_3^-$ ,  $g_2g_3^-$ ,  $g_1g_2^-$ ,  $g_1g_2g_3^-$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. Let  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$  be any twisted intertwining operators of types  $\binom{W_4}{W_1 M_1}$ ,  $\binom{M_1}{W_2 W_3}$ ,  $\binom{W_4}{M_2 W_3}$ ,  $\binom{M_2}{W_1 W_2}$ , respectively.
- (a)  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies the convergence and extension property for products.
- (b)  $(\mathcal{Y}_3, \mathcal{Y}_4)$  satisfies the convergence and extension property for iterates.
- (c) There exists  $N \in \mathbb{N}$  such that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) = 0$ , for any  $n \in \mathbb{C}$  with  $\Re(n) < -N$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , and  $(\zeta_1, \zeta_2) \in F_2(\mathbb{C}^\times)$ .
- (See (9.4) for the definition of  $\tilde{\lambda}_n^{(2)}$ .)

Then, the associativity of twisted intertwining operators for  $\mathcal{GM}_{gr}(G)$  holds.

The associativity of twisted intertwining operators is equivalent to the existence of a canonical associativity isomorphism for the  $P(z)$ -tensor product of twisted modules. To be specific, a lemma is the following:

**Lemma 1.3.5.** *Suppose  $\mathcal{C}$  is a category of twisted  $V$ -modules such that for any  $z \in \mathbb{C}^\times$ , and any  $W_1$  and  $W_2$  in  $\mathcal{C}$ , the  $P(z)$ -tensor product  $W_1 \boxtimes_{P(z)} W_2$  of  $W_1$  and  $W_2$  in  $\mathcal{C}$  exists. The followings are equivalent:*

1. The category  $\mathcal{C}$  satisfies the associativity of twisted intertwining operators.
2. For any  $z_1, z_2 \in \mathbb{C}$  satisfying (1.32), there exists a unique natural isomorphism

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : - \boxtimes_{P(z_1)} (- \boxtimes_{P(z_2)} -) \Rightarrow (- \boxtimes_{P(z_1-z_2)} -) \boxtimes_{P(z_2)} -$$

such that for any twisted modules  $W_1, W_2$ , and  $W_3$  in  $\mathcal{C}$ , the map

$$\overline{\left( \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} \right)}_{W_1, W_2, W_3} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$$

$$\rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

satisfies that

$$\begin{aligned} & \overline{\left( \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} \right)_{W_1, W_2, W_3}} (w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = \\ & (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3, \text{ for any } w_i \in W_i, i = 1, 2, 3. \end{aligned} \quad (1.34)$$

As results of Lemma 1.3.5, Theorem 1.3.1, and Corollary 1.3.4, we have the following corollaries:

**Corollary 1.3.6.** *Assume  $\mathcal{C}$  is a category of twisted modules satisfying all the conditions in Theorem 1.3.1, or 1.3.2, or 1.3.3. View  $-\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-)$  and  $(-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$  as two functors from  $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$  to  $\mathcal{C}$ . Then, for any  $z_1, z_2 \in \mathbb{C}$  satisfying (1.32), there exists a unique natural isomorphism*

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : -\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-) \Rightarrow (-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$$

satisfying (1.34).

**Corollary 1.3.7.** *Let  $G$  be a subgroup of  $\text{Aut}(V)$ . Assume all conditions in Corollary 1.3.4 are satisfied. View  $-\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-)$  and  $(-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$  as two functors from  $\mathcal{GM}_{gr}(G) \times \mathcal{GM}_{gr}(G) \times \mathcal{GM}_{gr}(G)$  to  $\mathcal{GM}_{gr}(G)$ . Then, for any  $z_1, z_2 \in \mathbb{C}$  satisfying (1.32), there exists a unique natural isomorphism*

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : -\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-) \Rightarrow (-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$$

satisfying (1.34).

## Chapter 2

# Twisted Modules

We recall in this section the notions of (generalized) twisted module for a vertex operator algebra.

Let  $(V, Y, \omega, \mathbf{1})$  be a vertex operator algebra. Let  $g$  be an automorphism of  $V$ . We first recall the definition of generalized  $g$ -twisted  $V$ -module first introduced in [H8]. In particular, in this paper, the vertex operator map for a  $g$ -twisted  $V$ -module in general contain the logarithm of the variable and the operator  $L(0)$  in general does not have to act semisimply.

**Definition 2.0.1.** A  $g$ -twisted generalized  $V$ -module (without a  $g$ -action) is a  $\mathbb{C}$ -graded vector space  $W = \coprod_{n \in \mathbb{C}} W_{[n]}$  (graded by *weights*) equipped with a linear map

$$Y_W^g : V \otimes W \rightarrow W\{x\}[\log x],$$

$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n \in \mathbb{C}} \sum_{k=0}^K (Y_W^g)_{n,k}(v)(w)x^{-n-1}(\log x)^k$$

satisfying the following conditions:

1. The *equivariance property*: For  $p \in \mathbb{Z}$ ,  $z \in \mathbb{C}^\times$ ,  $v \in V$  and  $w \in W$ ,

$$(Y_W^g)^{p+1}(gv, z)w = (Y_W^g)^p(v, z)w,$$

where for  $p \in \mathbb{Z}$ ,  $(Y_W^g)^p(v, z)$  is the  $p$ -th analytic branch of  $Y_W^g(v, x)$ .

2. The *identity property*: For  $w \in W$ ,  $Y_W^g(\mathbf{1}, x)w = w$ .
3. The *duality property*: For any  $u, v \in V$ ,  $w \in W$  and  $w' \in W'$ , there exists a multivalued analytic function of the form

$$f(z_1, z_2) = \sum_{i,j,k,l=0}^N a_{ijkl} z_1^{m_i} z_2^{n_j} (\log z_1)^k (\log z_2)^l (z_1 - z_2)^{-t} \quad (2.1)$$

for  $N \in \mathbb{N}$ ,  $m_1, \dots, m_N, n_1, \dots, n_N \in \mathbb{C}$  and  $t \in \mathbb{N}$ , such that the series

$$\langle w', (Y_W^g)^p(u, z_1)(Y_W^g)^p(v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', (Y_W^g)^p(u, z_1)\pi_n(Y_W^g)^p(v, z_2)w \rangle,$$

$$\langle w', (Y_W^g)^p(v, z_2)(Y_W^g)^p(u, z_1)w \rangle = \sum_{n \in \mathbb{C}} \langle w', (Y_W^g)^p(v, z_2)\pi_n(Y_W^g)^p(u, z_1)w \rangle,$$

$$\langle w', (Y_W^g)^p(Y_V(u, z_1 - z_2)v, z_2)w \rangle = \sum_{n \in \mathbb{C}} \langle w', (Y_W^g)^p(\pi_n Y_V(u, z_1 - z_2)v, z_2)w \rangle$$

are absolutely convergent in the regions  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$ ,  $|z_2| > |z_1 - z_2| > 0$ , respectively, and their sums are equal to the branch

$$f^{p,p}(z_1, z_2) = \sum_{i,j,k,l=0}^N a_{ijkl} e^{m_i l_p(z_1)} e^{n_j l_p(z_2)} l_p(z_1)^k l_p(z_2)^l (z_1 - z_2)^{-t}$$

of  $f(z_1, z_2)$  in the region  $|z_1| > |z_2| > 0$ , the region  $|z_2| > |z_1| > 0$ , the region given by  $|z_2| > |z_1 - z_2| > 0$  and  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ , respectively.

4. The  $L(0)$ -grading condition: Let  $L_W^g(0) = (Y_W^g)_{1,0}(\omega)$ . Then for  $n \in \mathbb{C}$ ,  $w \in W_{[n]}$ , there exists  $K \in \mathbb{Z}_+$  such that  $(L_W^g(0) - n)^K w = 0$ .
5. The  $L(-1)$ -derivative property: For  $v \in V$ ,

$$\frac{d}{dx} Y_W^g(v, x) = Y_W^g(L_V(-1)v, x).$$

**Definition 2.0.2.** A  $g$ -twisted generalized  $V$ -module **with a  $g$ -action** is a  $\mathbb{C} \times \mathbb{C}/\mathbb{Z}$ -graded vector space  $W = \coprod_{n \in \mathbb{C}, \alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$  (graded by *weights and  $g$ -weights*) equipped with a linear map

$$Y_W^g : V \otimes W \rightarrow W\{x\}[\log x],$$

$$v \otimes w \mapsto Y_W^g(v, x)w = \sum_{n \in \mathbb{C}} \sum_{k=0}^K (Y_W^g)_{n,k}(v)(w)x^{-n-1}(\log x)^k,$$

and a  $\langle g \rangle$ -module structure (where  $\langle g \rangle$  denote the cyclic group generated by  $g$ ), satisfying the following conditions:

1. Write  $W_{[n]} = \coprod_{\alpha \in \mathbb{C}/\mathbb{Z}} W_{[n]}^{[\alpha]}$ . Then,  $W = \coprod_{n \in \mathbb{C}} W_{[n]}$  equipped with  $Y_W^g$  is a  $g$ -twisted generalized  $V$ -module without a  $g$ -action.
2. The  $g$ -grading condition: Write  $W^{[\alpha]} = \coprod_{n \in \mathbb{C}} W_{[n]}^{[\alpha]}$ . For  $\alpha \in \mathbb{C}/\mathbb{Z}$ ,  $w \in W^{[\alpha]}$ , there exists  $\Lambda \in \mathbb{N}$ , such that  $(g - e^{2\pi i \alpha})^\Lambda w = 0$ .
3. The  $g$ -compatibility condition: For  $u \in V$ ,  $gY_W^g(u, x) = Y_W^g(gu, x)g$ . In other words, the action of  $g$  on  $W$  is a module automorphism.

A generalized twisted module is said to be *ordinary* if  $L_W^g(0)$  is diagonalizable. Since modules that are considered are all generalized, we will omit the word ‘generalized’ throughout this work.

A *lower-bounded  $g$ -twisted  $V$ -module* (with or without a  $g$ -action) is a  $g$ -twisted  $V$ -module  $W$  (with or without a  $g$ -action) such that for each  $n \in \mathbb{C}$ ,  $W_{[n+l]} = 0$  for sufficiently negative real number  $l$ . A  $g$ -twisted  $V$ -module  $W$  (with or without a  $g$ -action) is said to be *grading-restricted* if it is lower bounded and for each  $n \in \mathbb{C}$ ,  $\dim W_{[n]} < \infty$ .

For a subgroup  $G \leq \text{Aut}(V)$ , let  $\mathcal{GM}_{gr}(G)$  be the category of grading-restricted  $g$ -twisted  $V$ -modules for  $g \in G$ . Let  $\mathcal{GM}_{gr}$  be the category  $\mathcal{GM}_{gr}(\text{Aut}(V))$ .

For simplicity, we shall sometimes omit the subscript  $W$  and the superscript  $g$ . For example, we denote the twisted vertex operator map  $Y_W^g$  by  $Y^g$ ,  $Y_W$  or even  $Y$ , the Virasoro action  $L_W^g(n)$  by  $L(n)$  for  $n \in \mathbb{Z}$ .

Throughout this work, the phrase ‘without a  $g$ -action’ can be omitted. In other words, a  $g$ -twisted  $V$ -module refers to a  $g$ -twisted  $V$ -module without a  $g$ -action. When we refer to a  $g$ -twisted  $V$ -module with a  $g$ -action, the phrase ‘with a  $g$ -action’ will be stated explicitly.

Without otherwise specified,  $W, W_1, W_2, W_3$  are  $g$ -,  $g_1$ -,  $g_2$ -,  $g_3$ -twisted  $V$ -modules (i.e., twisted  $V$ -modules without group actions).

**Remark 2.0.3.** *The following fact looks obvious but actually need a proof. Let  $x_1, x_2$  be formal variables. Then,*

$$\langle w', Y_W^g(u, x_1) Y_W^g(v, x_2) w \rangle = \sum_{i,j,k,l=0}^N a_{ijkl} x_1^{m_i} x_2^{n_j} (\log x_1)^k (\log x_2)^l \cdot (\iota_{12}(x_1 - x_2)^{-1})^t, \quad (2.2)$$

where  $\iota_{12}(x_1 - x_2)^{-1} = \sum_{k \in \mathbb{N}} x_1^{-k-1} x_2^k$ . Without loss of generality, we can assume  $u \in V^{[a+\mathbb{Z}]}$ ,  $v \in V^{[b+\mathbb{Z}]}$ ,  $a, b \in \mathbb{C}^\times$ . Then we know that  $Y_W^g(u, x_1) \in x_1^{-a} \text{End}(W)[[x_1, x_1^{-1}]][\log x_1]$ , and  $Y_W^g(v, x_2) \in x_2^{-b} \text{End}(W)[[x_2, x_2^{-1}]][\log x_2]$ . Then, (2.2) relies on not only the duality but also the fact that

$$\begin{aligned} & ((-a + \mathbb{Z}) \cup \cup_{i=1}^N (m_i + \mathbb{Z})) \times \{0, \dots, N'\}, \\ & ((-b + \mathbb{Z}) \cup \cup_{i=1}^N (n_i + \mathbb{Z})) \times \{0, \dots, N'\} \end{aligned}$$

are unique expansion sets, where  $N'$  is  $N$  plus the largest possible power of  $\log x$  in  $Y_W^g(u, x)$  and  $Y_W^g(v, x)$  (See [HLZ5] for the definition of unique expansion set).

From the duality, Remark 2.0.3, and the fact that  $Y_W^g(\omega, x) \in \text{End}(W)[[x, x^{-1}]]$ , using Cauchy formula (see Proposition 7.0.10 and Corollary 7.0.12), we can prove the

commutator formula and therefore the Virasoro-bracket relation,

$$[L(m), Y_W^g(v, x)] = \sum_{k \in \mathbb{N}} \binom{m+1}{k} x^{m-k+1} Y_W^g(L(k-1)v, x), \text{ for any } m \in \mathbb{Z}, v \in V. \quad (2.3)$$

In particular, letting  $m = 0$ , the  $L(0)$ -bracket formula holds. Therefore, Proposition 3.20 in [HLZ2] for  $\mathcal{Y} = Y_W^g$  still holds here, which means the  $L(0)$ -grading is compatible with the action of components of  $Y_W^g$ , i.e. for any  $n \in \mathbb{C}$ ,  $k \in \mathbb{N}$ , and homogeneous  $v \in V, w \in W$ ,

$$\text{wt}((Y_W^g)_{n,k}(v)(w)) = \text{wt } v + \text{wt } w - n - 1. \quad (2.4)$$

**Remark 2.0.4.** *In Definition 2.0.1, we do not require a usually necessary axiom, which is the lower truncation property. A  $g$ -twisted  $V$ -module  $(W, Y_W^g)$  is said to be lower truncated if*

*for any  $v \in V, w \in W, n \in \mathbb{C}$  and  $k \in \mathbb{N}$ ,*

$$(Y_W^g)_{n+l,k}(v)(w) = 0, \text{ for sufficiently large } l \in \mathbb{N}. \quad (2.5)$$

*For those versions of definition of module using Jacobi identity, lower truncation property is needed to make sure that the Jacobi identity is well-defined. However, if one uses duality to define module, the lower truncation property does not have to be in the definition any longer.*

*Lower truncation property is a weaker condition than lower-bounded. By (2.4), any lower-bounded twisted module is automatically lower truncated. Moreover, by (2.4), a lower-bounded twisted module is actually fully truncated. (See Assumption 4.0.1)*

**Remark 2.0.5.** *By letting  $v = \omega$  in (2.3), we know that any twisted  $V$ -module  $W$*

is a module for Virasoro algebra  $\mathcal{L}$ . Since  $L_n \mapsto L_{-n}$  is an anti-automorphism of Virasoro algebra,  $W'$  has a natural  $\mathcal{L}$ -module structure given by

$$\langle L(n)_{W'} w', w \rangle = \langle w', L(-n)_W w \rangle, \quad n \in \mathbb{Z}, w \in W, w' \in W'.$$

Let  $(W, Y_W^g)$  be a  $g$ -twisted  $V$ -module. Let  $h$  be an automorphism of  $V$  and let

$$\begin{aligned} \phi_h(Y_W^g) : V \otimes W &\rightarrow W\{x\}[\log x] \\ v \otimes w &\mapsto \phi_h(Y^g)(v, x)w \end{aligned}$$

be the linear map defined by

$$\phi_h(Y_W^g)(v, x)w = Y_W^g(h^{-1}v, x)w.$$

Then we have:

**Proposition 2.0.6.** *The pair  $(W, \phi_h(Y_W^g))$  is an  $hgh^{-1}$ -twisted  $V$ -module. Moreover, when  $W$  is a module with  $g$ -action, for any  $n \in \mathbb{Z}$ , the map  $w \mapsto g^{-n}w$  is an isomorphism between  $(W, Y_W^g)$  and  $(W, \phi_{g^n}(Y_W^g))$  (as  $g$ -twisted  $V$ -modules with  $g$ -action).*

We shall denote the  $hgh^{-1}$ -twisted  $V$ -module in the proposition above by  $\phi_h(W)$ .

Note that when  $h = g$ , we obtain a  $g$ -twisted module  $\phi_g(W)$  for which the twisted vertex operator is given by

$$\phi_g(Y_W^g)(v, x)w = Y_W^g(g^{-1}v, x)w.$$

But this  $g$ -twisted  $V$ -module is equivalent to the original  $g$ -twisted module  $W$  if  $W$  has a  $g$ -action. The equivalence is given by  $g^{-1} : W \rightarrow W$  since we have

$$g^{-1}Y_W^g(v, x)g = Y_W^g(g^{-1}v, x) = \phi_g(Y_W^g)(v, x)$$

for  $v \in V$ . By the equivariance property, we have  $(Y_W^g)^p(g^{-1}v, x) = (Y_W^g)^{p+1}(v, x)$  and, if

$$Y_W^g(v, x)w = \sum_{k=0}^K \sum_{n \in \mathbb{C}} (Y_W^g)_{-n-1, k}(v) x^n (\log x)^k$$

for  $v \in V$  and  $w \in W$ , we have

$$Y_W^g(g^{-1}v, x)w = \sum_{k=0}^K \sum_{n \in \mathbb{C}} (Y_W^g)_{-n-1, k}(v) e^{2\pi i n} x^n (\log x + 2\pi i)^k.$$

We also need contragredient twisted  $V$ -modules. Let  $(W, Y_W^g)$  be a  $g$ -twisted  $V$ -module. Let  $W'$  be the graded dual of  $W$ . Define a linear map

$$\begin{aligned} (Y_W^g)' : V \otimes W' &\rightarrow W'\{x\}[\log x], \\ v \otimes w' &\mapsto (Y_W^g)'(v, x)w' \end{aligned}$$

by

$$\langle (Y_W^g)'(v, x)w', w \rangle = \langle w', Y_W^g(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \rangle$$

for  $v \in V$ ,  $w \in W$  and  $w' \in W'$ .

**Proposition 2.0.7.** *Let  $L_W^g(0)_N$  be the nilpotent part of  $L_W^g(0)$ . Suppose  $L_W^g(0)_N$  is nilpotent on  $W$  (not just locally nilpotent). Then, the pair  $(W', (Y_W^g)')$  is a  $g^{-1}$ -twisted  $V$ -module.*

The proof of this result is a special case of the proof of Theorem 5.0.1 in Section 4 with  $W_1 = V$ ,  $g_1 = 1_V$ ,  $g_2 = g$ ,  $W_2 = W_3 = W$ , and  $\mathcal{Y} = Y_W$ . Since the proof of Theorem 5.0.1 uses only the definition of  $(Y_W^g)'$ , quoting the proof of Theorem 5.0.1 to give a proof of Proposition 2.0.7 does not constitute circular reasoning.

The  $g^{-1}$ -twisted  $V$ -module  $(W', (Y_W^g)')$  is called the *contragredient module* of  $(W, Y_W^g)$ .

# Chapter 3

## Twisted Intertwining Operators

In this chapter, we introduce a notion of twisted intertwining operators which is more general than the one in [H9]. We also give the basic results on such twisted intertwining operators.

### 3.0.1 Multivalued functions on configuration spaces

For any set  $X$  and  $n \in \mathbb{Z}_+$ , define  $F_1(X) = X$ , and  $F_n(X) = \{(z_1, \dots, z_n) \in X^n \mid z_i \neq z_j \text{ for any } i \neq j\}$  for  $n > 1$ . It's easy to see that  $F_n$  can be an endofunctor on some categories, for example, the category of sets, topological spaces, smooth manifolds, complex manifolds and so on. Let  $F_{1,1}^0(\mathbb{C}^\times)$  be the topological space given by cutting  $F_2(\mathbb{C}^\times)$  along the positive real lines in the  $a_1$ -,  $a_2$ - and  $a_1 - a_2$ -planes, that is, the sets

$$\{(a_1, a_2) \in F_2(\mathbb{C}^\times) \mid a_1 \in \mathbb{R}_+\},$$

$$\{(a_1, a_2) \in F_2(\mathbb{C}^\times) \mid a_2 \in \mathbb{R}_+\},$$

$$\{(a_1, a_2) \in F_2(\mathbb{C}^\times) \mid a_1 - a_2 \in \mathbb{R}_+\},$$

with these sets attached to the upper half  $a_1$ -,  $a_2$ - and  $a_1 - a_2$ -planes. More explicitly,  $F_{1,1}^0(\mathbb{C}^\times)$  is the topological space which set-theoretically is equal to  $F_2(\mathbb{C}^\times)$ . The

collection of open sets of  $F_{1,1}^0(\mathbb{C}^\times)$  is the minimal collection which satisfies the open sets axioms and contains

- open sets of  $F_2(\mathbb{C}^\times)$  in the standard metric space topology,
- the set  $\{(a_1, a_2) \in F_2(\mathbb{C}^\times) \mid \Re(a_1) > 0, \Im(a_1) \geq 0\}$ ,
- the set  $\{(a_1, a_2) \in F_2(\mathbb{C}^\times) \mid \Re(a_2) > 0, \Im(a_2) \geq 0\}$ ,
- the set  $\{(a_1, a_2) \in F_2(\mathbb{C}^\times) \mid \Re(a_1 - a_2) > 0, \Im(a_1 - a_2) \geq 0\}$ .

Clearly,  $F_{1,1}^0(\mathbb{C}^\times)$  has a unique structure of *complex manifold with boundary* such that the set-theoretically identity map  $F_{1,1}^0(\mathbb{C}^\times) \rightarrow F_2(\mathbb{C}^\times)$  is holomorphic.

For any  $(z_1, z_2) \in F_{1,1}^0(\mathbb{C}^\times)$ , choose a continuous path  $\gamma$  in  $F_{1,1}^0(\mathbb{C}^\times)$  from  $(z_1, z_2)$  to  $(-3, -2)$ . Such a path exists and is unique up to boundary-fixing homotopy due to the fact that  $F_{1,1}^0(\mathbb{C}^\times)$  is path-connected and simply connected. The path  $\gamma$  induces an isomorphism  $\pi_1(F_2(\mathbb{C}^\times), (z_1, z_2)) \rightarrow \pi_1(F_2(\mathbb{C}^\times), (-3, -2))$ , which is independent on the choice of  $\gamma$ . Using this isomorphism, we can identify elements in  $\pi_1(F_2(\mathbb{C}^\times), x)$  for different  $x \in F_2(\mathbb{C}^\times)$ . Therefore, in the following, we would define  $b_{12}, b_{13}$  and  $b_{23}$  as three elements in  $\pi_1(F_2(\mathbb{C}^\times), (-3, -2))$ . But  $b_{12}, b_{13}, b_{23}$  also can be viewed as elements in  $\pi_1(F_2(\mathbb{C}^\times), x)$  for arbitrary  $x \in F_2(\mathbb{C}^\times)$ .

Let  $b_{12}$  be the element in  $\pi_1(F_2(\mathbb{C}^\times), (-3, -2))$  represented by the loop

$$[0, 1] \ni t \mapsto (-2 + e^{2\pi it + \pi}, -2).$$

Let  $b_{13}$  be the element in  $\pi_1(F_2(\mathbb{C}^\times), (-3, -2))$  represented by the loop

$$[0, 3] \ni t \mapsto \begin{cases} (-1 + 2e^{\pi it + \pi}, -2), & 0 \leq t \leq 1, \\ (e^{\pi i(t-1)}, -2), & 1 \leq t \leq 2, \\ (-2 + e^{-\pi i(t-2)}, -2), & 2 \leq t \leq 3. \end{cases}$$

Let  $b_{23}$  be the element in  $\pi_1(F_2(\mathbb{C}^\times), (-3, -2))$  represented by the loop

$$[0, 1] \ni t \mapsto (-3, 2e^{2\pi it + \pi}).$$

It can be check that

$$b_{13}b_{12}b_{23} = b_{12}b_{23}b_{13} = b_{23}b_{13}b_{12}. \quad (3.1)$$

Note that  $F_2(\mathbb{C}^\times)$  is homotopically equivalent to the configuration space  $F_3(\mathbb{C})$ . So, the fundamental group of  $F_2(\mathbb{C}^\times)$  is in fact the pure braid group  $PB_3 = \pi_1(F_3(\mathbb{C}))$ . We know that  $F_2(\mathbb{C}^\times)$  is isomorphic to the group defined using the generators  $b_{12}, b_{13}, b_{23}$  and the relations (3.1).

**Definition 3.0.1.** Let  $M$  be a complex manifold (with or without boundary). A *multivalued analytic function* on  $M$  is a collection  $(A_x)_{x \in M}$ , where  $A_x$  is a subset of the stalk of the holomorphic function sheaf on  $M$  at  $x$ . such that

- For any continuous path  $\gamma : [0, 1] \rightarrow M$ , and  $\psi \in A_{\gamma(0)}$ , the analytic continuation of  $\psi$  along  $\gamma$  exists. Moreover, the germ at  $\gamma(1)$  represented by this analytic continuation is contained in  $A_{\gamma(1)}$ .
- For any  $x, y \in M$ , and  $\psi_x \in A_x, \psi_y \in A_y$ , there exists a continuous path  $\gamma$  such that  $\psi_y$  can be obtained by doing the analytic continuation of  $\psi_x$  along  $\gamma$ .

Equivalently, a multivalued analytic function on  $M$  is a path-connected component of the étalé space of the holomorphic function sheaf on  $M$ .

**Definition 3.0.2.** A *maximally extended single-valued branch* of a multivalued analytic function  $(A_x)_{x \in F_2(\mathbb{C}^\times)}$  on  $F_2(\mathbb{C}^\times)$ , is a (single-valued) analytic function  $\phi$  on  $F_{1,1}^0(\mathbb{C}^\times)$ , such that for any  $x \in F_2(\mathbb{C}^\times)$ , the germ at  $x$  represented by  $\phi$  is contained in  $A_x$ .

Let  $f(z_1, z_2)$  be a multivalued function on  $F_2(\mathbb{C}^\times)$ , and let  $f^e(z_1, z_2)$  be a maximally extended single-valued branch of  $f(z_1, z_2)$ . For any loop  $\gamma : [0, 1] \rightarrow F_2(\mathbb{C}^\times)$ , we can do the analytic continuation of  $f^e(z_1, z_2)$  along  $\gamma$  and obtain another maximally extended single-valued branch of  $f(z_1, z_2)$ . Since we know that the maximally extended single-valued branch we obtain only depends on the homotopy class of  $\gamma$ , we denote this maximally extended single-valued branch by

$$f^{[\gamma]}(z_1, z_2),$$

where  $[\gamma]$  is the element in  $\pi_1(F_2(\mathbb{C}^\times))$ . In this way, we get a right action of the group  $\pi_1(F_2(\mathbb{C}^\times))$  on the set of maximally extended single-valued branches of  $f(z_1, z_2)$ , defined by

$$g \cdot (f^e(z_1, z_2)) = f^g(z_1, z_2), \quad g \in \pi_1(F_2(\mathbb{C}^\times)). \quad (3.2)$$

It is not hard to check that this is a transitive right action, i.e.,

- Any maximally extended single-valued branches of  $f(z_1, z_2)$  is equal to  $f^g(z_1, z_2)$  for some  $g \in \pi_1(F_2(\mathbb{C}^\times))$ .
- $e \cdot (f^g(z_1, z_2)) = f^g(z_1, z_2)$ , for any  $g \in \pi_1(F_2(\mathbb{C}^\times))$ , where  $e$  is the identity element in the group  $\pi_1(F_2(\mathbb{C}^\times))$ .
- $(g_1 g_2) \cdot (f^g(z_1, z_2)) = f^{g_1 g_2 g}(z_1, z_2) = g_2 \cdot (g_1 \cdot f^g(z_1, z_2))$ , for any  $g, g_1, g_2 \in \pi_1(F_2(\mathbb{C}^\times))$ .

**Remark 3.0.3.** *The above definitions and the group action can be generalized to general complex manifolds, especially some more complicated configuration spaces. For any complex manifold, one can do cuttings along some (real) codimension 1 submanifolds with boundaries such that the original manifold remains connected and becomes simply connected. Once fixing such a cutting, one can define the notion of maximally*

extended single-valued branch, and obtain the right action of the fundamental group on the set of maximally extended single-valued branches of a multivalued function.

**Definition 3.0.4.** For any complex manifold  $M$  where we have defined the notion of maximally extended single-valued branch of a multivalued function on  $M$  (see Remark 3.0.3), a *multivalued function with a preferred (maximally extended single-valued) branch* is a pair  $(f, f^e)$  where  $f$  is a multivalued function on  $M$ , and  $f^e$  is a maximally extended single-valued branch of  $f$ .

**Remark 3.0.5.** For example, we can define the notion of a multivalued function with a preferred branch on  $F_{k,l}$  (see Condition 3.0.13), since the space  $F_{k,l}^1$  has been defined (see the two paragraphs following Condition 3.0.13).

**Remark 3.0.6.** We cannot add two multivalued function together (we can do scalar multiplication tho). However, the space of multivalued functions with a preferred branch form a commutative associative algebra over  $\mathbb{C}$ , because we can do linear combination of the preferred maximally extended single-valued branches.

**Definition 3.0.7.** Let  $\delta = (\delta_1, \dots, \delta_n)$ , where  $\delta_i \in \{0, \infty\}$ ,  $i = 1, \dots, n$ . Suppose  $f(z_1, \dots, z_n)$  is a multi-valued analytic function defined on an open region  $\Omega$  of  $\mathbb{C}^n$ . We say  $(z_1, \dots, z_n) = \delta$  is a *component-isolated singularity* of  $f(z_1, \dots, z_n)$  if there exists  $r \in \mathbb{R}_+^n$  such that  $\Delta_{I_n, 0, \delta}^\times(r) \subset \Omega$ , where  $\Delta_{I_n, 0, \delta}^\times(r)$  is defined in Appendix A, (A.1). Let  $A \in \text{GL}(n, \mathbb{C})$  and  $\beta \in \mathbb{C}^n$  (written as a row vector). Then  $\zeta_1, \dots, \zeta_n$  given by

$$(\zeta_1, \dots, \zeta_n) = (z_1, \dots, z_n)A - \beta$$

are also independent variables. Define

$$g(\zeta_1, \dots, \zeta_n) = f((\zeta_1, \dots, \zeta_n)A^{-1} + \beta A^{-1}). \quad (3.3)$$

For  $\delta \in (\mathbb{C} \cup \{\infty\})^n$ , we say that  $(\zeta_1, \dots, \zeta_n) = \delta$  is a *component-isolated singularity*

of  $f(z_1, \dots, z_n)$  if  $(\zeta_1, \dots, \zeta_n) = \delta$  is a component-isolated singularity of  $g(\zeta_1, \dots, \zeta_n)$ .

**Remark 3.0.8.** Notice that  $(\zeta_1, \dots, \zeta_n) = (z_1, \dots, z_n)A - \beta = \delta$  being a component-isolated singularity of a function is not equivalent to  $(z_1, \dots, z_n) = \delta A^{-1} + \beta A^{-1}$  being a component-isolated singularity of the same function. This is because we have different sets of independent variables. For example, consider  $f(z_1, z_2) = \frac{1}{z_1 - z_2}$ . Since  $(\zeta_1, \zeta_2) = (0, 0)$  is a component-isolated singularity of the function  $g(\zeta_1, \zeta_2) = 1/\zeta_1$ , we also say that  $(z_1 - z_2, z_2) = (0, 0)$  is a component-isolated singularity of  $f(z_1, z_2)$ . In this case,  $z_1 - z_2$  and  $z_2$  are independent variables. However,  $(z_1, z_2) = (0, 0)$  is clearly not a component-isolated singularity of  $f(z_1, z_2)$ . In this case, the independent variables are  $z_1$  and  $z_2$ .

We say that two component-isolated singularities  $(z_1, \dots, z_n)A - \beta = \delta$  and  $(z_1, \dots, z_n)A' - \beta' = \delta'$  are *equivalent* if there exists a matrix  $U$  in the subgroup of  $\text{GL}_n(\mathbb{C})$  generated by all permutation matrices and invertible diagonal matrices such that  $A' = AU$ ,  $\beta' = \beta U$  and  $\delta' = \delta U$ .

**Example 3.0.9.** Let  $S \subset \mathbb{C}$  be a non-empty finite set. Let  $A \in \text{GL}_n(\mathbb{C})$ ,  $\beta \in \mathbb{C}^n$ ,  $\delta = (\delta_1, \dots, \delta_n) \in \{0, \infty\}^n$ . Let  $I_\infty = \{i \mid \delta_i = \infty\}$ ,  $(\tilde{\beta}_1, \dots, \tilde{\beta}_n) = \beta A^{-1}$ , and  $A^{-1} = (b_{ij})$ .

• Let  $f(z_1, \dots, z_n)$  be a multi-valued analytic function on  $F_n(\mathbb{C} - S) \subset \mathbb{C}^n$ . Then the component-isolated singularities of  $f(z_1, \dots, z_n)$  are the followings.

- $I_\infty = \emptyset$ .  $\tilde{\beta}_1, \dots, \tilde{\beta}_n$  are all distinct. For any  $j$  with  $\tilde{\beta}_j \in S$ , the  $j$ -th column of  $A^{-1}$  has only one nonzero entries. Then,  $(z_1, \dots, z_n)A - \beta = \delta$  is a component-isolated singularity.
- $I_\infty = \{i_0\}$ . Let  $J_\infty = \{j \mid b_{i_0 j} \neq 0\}$ ,  $J_0 = \{1, \dots, n\} - J_\infty$ . Then  $b_{i_0 j}$  are all equal for  $j \in J_\infty$ . For  $j \in J_0$ ,  $\tilde{\beta}_j$  are all distinct. For  $j \in J_\infty$ ,  $\tilde{\beta}_j$  are all distinct. For any  $j \in J_0$ , and  $\tilde{\beta}_j \in S$ , the  $j$ -th column of  $A^{-1}$  has only one nonzero entry. Then,  $(z_1, \dots, z_n)A - \beta = \delta$  is a component-isolated singularity.

• Let  $f(z_1, \dots, z_n)$  be a multi-valued analytic function on  $(\mathbb{C} - S)^n \subset \mathbb{C}^n$ . Then  $(z_1, \dots, z_n)A - \beta = \delta$  is a component-isolated singularity if and only if the followings are satisfied. For any  $j$ ,  $\#I_\infty \cap \{i | b_{ij} \neq 0\} \leq 1$ . For those  $j$  with  $\#I_\infty \cap \{i | b_{ij} \neq 0\} = 0$  and  $\tilde{\beta}_j \in S$ , the  $j$ -th column of  $A^{-1}$  has only one nonzero entry.

**Definition 3.0.10.** Let  $f(z_1, \dots, z_n)$  be a multi-valued analytic function defined on an open region of  $\mathbb{C}^n$ . Let  $\delta = (\delta_1, \dots, \delta_n) \in \{0, \infty\}^n$ . Suppose  $(z_1, \dots, z_n) = \delta$  is a component-isolated singularity of  $f(z_1, \dots, z_n)$ . Let  $\{f^b(z_1, \dots, z_n)\}_{b \in B}$  be the set of all single valued branches of  $f(z_1, \dots, z_n)$  near  $\delta$  with cuts at  $z_i \in \mathbb{R}_+$ . Then for each  $b \in B$ , there exists  $r_b \in \mathbb{R}_+^n$  such that  $f^b(z_1, \dots, z_n)$  is analytic on  $\Delta_{I_n, 0, \delta}^\times(r_b)$ . We say that  $(z_1, \dots, z_n) = \delta$  is a *regular singularity* of  $f(z_1, \dots, z_n)$  if there exists  $K \in \mathbb{N}$ ,  $D_i = \cup_{j=1}^{N_i} r_j^{(i)} + \mathbb{N}$  (or  $D_i = \cup_{j=1}^{N_i} r_j^{(i)} - \mathbb{N}$ ) where  $r_1^{(i)}, \dots, r_{N_i}^{(i)} \in \mathbb{C}$  for  $\delta_i = 0$  (or  $\delta_i = \infty$ ), and  $\alpha_{a_1, j_1; \dots; a_n, j_n}^{(b)} \in \mathbb{C}$ , such that on the region  $\Delta_{I_n, 0, \delta}^\times(r_b)$ , the right-hand side of the following equation is absolutely convergent, and

$$f^b(z_1, \dots, z_n) = \sum_{i=1}^n \sum_{a_i \in D_i} \sum_{j_1, \dots, j_n=0}^K \alpha_{a_1, j_2; \dots; a_n, j_n}^{(b)} z_1^{a_1} (\log z_1)^{j_1} \cdots z_n^{a_n} (\log z_n)^{j_n}. \quad (3.4)$$

Let  $A \in \text{GL}(n, \mathbb{C})$  and  $\beta \in \mathbb{C}^n$  be the same as above. We say that  $(\zeta_1, \dots, \zeta_n) = \delta$  is a *regular singularity* of  $f(z_1, \dots, z_n)$  if  $(\zeta_1, \dots, \zeta_n) = \delta$  is a regular singularity of  $g(\zeta_1, \dots, \zeta_n)$ , where  $g(\zeta_1, \dots, \zeta_n)$  is give by (3.3).

**Remark 3.0.11.** We generalize the notion of component-isolated singularity and regularity singularity in the following sense. Let  $\delta \in \{0, \infty\}^n$ . Let  $\Omega$  be an open region in  $\mathbb{C}^n$ . Let  $f$  be a multi-valued analytic function defined on  $\Omega$ . Let  $\psi$  be a injective holomorphic map from  $U$  to  $\mathbb{C}^n$ , where  $U$  is an open subset of  $\Omega$ . We say  $\psi(z_1, \dots, z_n) = \delta$  is a component-isolated singularity, or regular singularity of  $f$  if  $(z_1, \dots, z_n) = \delta$  is a component-isolated singularity, or regular singularity of  $f \circ \psi^{-1}$ , respectively. If  $\psi(z_1, \dots, z_n) = (z_1, \dots, z_n)A - \beta$ , then it is consistent with Definitions 3.0.7 and 3.0.10.

**Remark 3.0.12.** Let  $I \subset \{1, \dots, n\}$ . Without Loss of Generality, assume  $I = \{1, \dots, m\}$ . Let  $\Omega$  be open regions in  $\mathbb{C}^n$  and let  $f$  be a multi-valued analytic function defined on  $\Omega$  of  $\mathbb{C}^n$ . Let  $\delta \in \{0, \infty\}^{n-m}$ . Let  $X$  be a subset of  $\mathbb{C}^m$ . Let  $\psi = (\psi_1, \dots, \psi_{n-m})$  be a holomorphic map from an open subset  $U$  of  $\Omega$  to  $\mathbb{C}^{n-m}$  satisfying:

1. The matrix  $\left( \frac{\partial \psi_i}{\partial z_{m+j}} \right)_{1 \leq i, j \leq n-m}$  is nondegenerate everywhere.
2. For any  $(\tilde{z}_1, \dots, \tilde{z}_m) \in X$ , the set  $\{(z_1, \dots, z_n) \in U \mid z_i = \tilde{z}_i, i = 1, \dots, m\}$  is nonempty.

By 1., we know that  $\psi$  restricted on the set in 2. is injective. Denote the inverse of this restriction of  $\psi$  by  $\psi_{(\tilde{z}_1, \dots, \tilde{z}_m)}^{-1}$ . We say that  $\psi(z_1, \dots, z_n) = \delta$  is a *component-isolated singularity, or regular singularity of  $f$  for  $(z_1, \dots, z_m) \in X$* , if  $(\zeta_{m+1}, \dots, \zeta_n) = \delta$  is a component-isolated singularity, or regular singularity of

$$f(\tilde{z}_1, \dots, \tilde{z}_m, \psi_{(\tilde{z}_1, \dots, \tilde{z}_m)}^{-1}(\zeta_{m+1}, \dots, \zeta_n)),$$

respectively, for all  $(\tilde{z}_1, \dots, \tilde{z}_m) \in X$ . Let  $Y$  be the image of  $\Omega$  under the projection  $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_m)$ . Then, we say that  $\psi(z_{m+1}, \dots, z_n) = \delta$  is a *component-isolated singularity, or regular singularity of  $f$*  if  $\psi(z_{m+1}, \dots, z_n) = \delta$  is a component-isolated singularity, or regular singularity of  $f$  for  $(z_1, \dots, z_m) \in Y$ .

Suppose that we have a multivalued function  $f$ , injective holomorphic map  $\psi = (\psi_{m+1}, \dots, \psi_n)$ , and open region  $X \subset \mathbb{C}^n$  such that  $\psi(z_1, \dots, z_n) = \delta$  is a regular singularity of  $f$  for  $(z_1, \dots, z_m) \in X$ . Then, on the region given by  $(z_1, \dots, z_n) \in \text{Domain}(f)$ ,  $(z_1, \dots, z_m) \in X$ ,  $(z_{m+1}, \dots, z_n) \in \psi_{(z_1, \dots, z_m)}^{-1}(\Delta_{I_{n-m}, 0, \delta}^\times(r))$  for some  $r \in \mathbb{R}_+^{n-m}$ , any single-valued branch of  $f$  can be written as an absolute convergent series:

$$\sum_{j=m+1}^n \sum_{b_j \in \mathbb{C}, k_j \in \mathbb{N}} a_{b_1, k_1 \dots b_m, k_m} (z_1, \dots, z_m) \psi_{m+1}(z_{m+1}, \dots, z_n)^{b_{m+1}} \log(\psi_{m+1}(z_{m+1}, \dots, z_n))^{k_{m+1}} \dots \psi_n(z_{m+1}, \dots, z_n)^{b_n} \log(\psi_n(z_{m+1}, \dots, z_n))^{k_n}. \quad (3.5)$$

As a generalization of the power series case, we can show that those coefficients  $a_{b_1, k_1 \dots b_m, k_m}(z_1, \dots, z_m)$  are holomorphic function on the region  $(z_1, \dots, z_m) \in X$ , which are possible to be single-valued branches of multivalued functions.

Let  $S \subset \mathbb{C}$  be a finite set containing 0. Let  $k, l \in \mathbb{N}$  with  $k + l > 0$ . Let  $n = k + l$ .

Let  $\mathcal{G}_{1,k,l}$  be the subgroup of  $\mathrm{GL}_n(\mathbb{C})$  generated by the permutation matrices  $(\delta_{i, \sigma(i)})$  for  $\sigma \in \mathrm{Sym}(n)$  with  $\sigma(\{1, \dots, k\}) = \{1, \dots, k\}$ . We know that  $\mathcal{G}_{1,k,l} \cong \mathrm{Sym}(k) \times \mathrm{Sym}(l)$ . Let  $\mathcal{G}_{2,k,l}$  be the subgroup of  $\mathrm{GL}_n(\mathbb{C})$  generated by the matrices  $U_j$  uniquely determined by  $(x_1, \dots, x_n)U_j = (x_1 - x_j, \dots, x_{j-1} - x_j, -x_j, x_{j+1} - x_j, \dots, x_n - x_j)$  for  $j = k+1, \dots, n$ . Here, the  $x_i$  are formal variables. More explicitly, we have

$$U_j = \left( \begin{array}{c|ccc} 1 & & & \text{j-th column} \\ & \ddots & & -1 \\ & & & \vdots \\ & & 1 & -1 \\ \hline & & 1 & -1 \\ & & \ddots & \vdots \\ & & & 1 & -1 \\ & & & & -1 \\ & & & & -1 & 1 \\ & & & & \vdots & \ddots \\ & & & & -1 & & 1 \end{array} \right). \quad (3.6)$$

For  $\sigma = (m'm) \in \mathrm{Sym}(n)$ , with  $m, m' = k+1, \dots, n$ ,  $m \neq m'$ , we know that  $U_{m'}^2 = 1$ , and that  $U_{m'}^{-1}U_m U_{m'}$  is a permutation matrix  $(\delta_{i, \sigma(j)})$ . Therefore,  $\mathcal{G}_{2,k,l} \cong \mathrm{Sym}(l+1)$  by  $U_j \mapsto (1, j - k + 1)$ . Let  $\mathcal{G}_{k,l}$  be the subgroup of  $\mathrm{GL}_n(\mathbb{C})$  generated by  $\mathcal{G}_{1,k,l}$  and  $\mathcal{G}_{2,k,l}$ . We know that  $\mathcal{G}_{2,k,l} \trianglelefteq \mathcal{G}_{k,l}$  (and therefore  $\mathcal{G}_{k,l} = \mathcal{G}_{1,k,l}\mathcal{G}_{2,k,l}$ ), and  $\mathcal{G}_{k,l} \cong \mathrm{Sym}(k) \times \mathrm{Sym}(l+1)$ . Also, we know that the collection  $\mathcal{G}_{1,k,l} \backslash \mathcal{G}_{k,l}$  of right cosets is equal to  $\{\mathcal{G}_{1,k,l}, \mathcal{G}_{1,k,l}U_{k+1}, \dots, \mathcal{G}_{1,k,l}U_{k+l}\}$ .

Now we define the following biholomorphic maps from  $F_n(\mathbb{C}^\times)$  to  $F_n(\mathbb{C}^\times)$ :

$$\begin{aligned} \psi_U &: F_n(\mathbb{C}^\times) \rightarrow F_n(\mathbb{C}^\times) \quad , \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_n)U, \quad \text{for any } U \in \mathcal{G}_{k,l}; \\ \theta_n &: F_n(\mathbb{C}^\times) \rightarrow F_n(\mathbb{C}^\times) \quad , \quad (z_1, \dots, z_n) \mapsto (z_1^{-1}, \dots, z_n^{-1}); \end{aligned} \quad (3.7)$$

Let  $\mathcal{S}_{k,l}$  be the group generated by  $\theta_{k+l}$  and  $\psi_U$  for all  $U \in \mathcal{G}_{k,l}$ . The group  $\mathcal{S}_{k,l}$  is infinite as long as  $l \neq 0$ . For any  $k, l \in \mathbb{N}$  with  $k+l \neq 0$ , we have the natural embedding

$$\begin{aligned} \mathcal{S}_{k,l} &\hookrightarrow \mathcal{S}_{k+1,l} & (3.8) \\ \theta_{k+l} &\mapsto \theta_{k+l+1} \\ \psi_{U_m}, \text{ where } U_m \in \mathcal{G}_{2,k,l} &\mapsto \psi_{U_m}, \text{ where } U_m \in \mathcal{G}_{2,k+1,l} \\ \psi_{(\delta_{i,\sigma(j)})}, \text{ where } \sigma \in \text{Sym}(k) &\mapsto \psi_{(\delta_{i,\sigma'(j)})}, \text{ where } \sigma' \in \text{Sym}(k+1) \text{ defined by} \\ &\sigma'(1) = 1, \sigma'(a) = \sigma(a-1) + 1, a = 2, \dots, k+1. \end{aligned}$$

The  $(\delta_{i,\sigma(j)})$  means the permutation matrix induced by a permutation  $\sigma$ .

Now we define a condition for a multivalued analytic function  $f(z_1, \dots, z_n)$  defined on  $F_n(\mathbb{C}-S)$ . We say that  $f(z_1, \dots, z_n)$  satisfies condition 3.0.13 for the triple  $(k, l, S)$  if the followings are satisfied:

**Condition 3.0.13.** (a) For a sufficiently large positive integer  $M$ , the function

$$\prod_{1 \leq i < j \leq k} (z_i - z_j)^M \cdot f(z_1, \dots, z_n)$$

can be analytically extended to a multivalued analytic function  $g(z_1, \dots, z_n)$  on the region  $F_{k,l}(\mathbb{C}-S)$  defined by

$$F_{k,l}(\mathbb{C}-S) = \{(z_1, \dots, z_n) \in (\mathbb{C}-S)^n \mid$$

$$z_i \neq z_j \text{ for any } i = k+1, \dots, n, j = 1, \dots, n, \text{ and } i \neq j\}. \quad (3.9)$$

(If  $k = 0$  or  $1$ ,  $f$  and  $g$  are the same multivalued analytic function.)

(b) The following component-isolated singularities of  $g(z_1, \dots, z_n)$  are regular:

- $(z_1, \dots, z_{k+l}) - \beta = \delta$ , for  $\beta = (\beta_1, \dots, \beta_{k+l}) \in S^{k+l}$ ,  $\delta = (\delta_1, \dots, \delta_{k+l}) \in \{0, \infty\}^{k+l}$ , such that (1)  $\beta_i = 0$  if  $\delta_i = \infty$  (2)  $\beta_i \neq \beta_j$  for any distinct  $i, j = 1, \dots, k+l$  with  $j > k$  and  $\delta_i = 0$ ; (3)  $\delta_i = 0$  if  $i > k$ . (This is empty if  $\#S < l$ .)
- $(z_1, \dots, z_{k+l}) - \beta = \delta$ , for  $\beta = (\beta_1, \dots, \beta_{k+l}) \in S^{k+l}$ ,  $\delta = (\delta_1, \dots, \delta_{k+l}) \in \{0, \infty\}^{k+l}$ , such that (1) there uniquely exists  $i_0 = 1, \dots, k+l$  such that  $\delta_{i_0} = \infty$ . Moreover,  $i_0 > k$  and  $\beta_{i_0} = 0$ ; (2)  $\beta_i \neq \beta_j$  for any distinct  $i, j = 1, \dots, k+l$  with  $j > k$ ,  $i \neq i_0$ , and  $j \neq i_0$ . (This is empty if either  $\#S < l$  or  $l = 0$ .)
- $(z_1 - z_i, \dots, z_{i-1} - z_i, z_i, z_{i+1}, \dots, z_k) = (0, \dots, 0, \infty, 0, \dots, 0)$  for any  $1 \leq i \leq k$ , if  $k \in \mathbb{Z}_+$ ,  $l = 0$ ,  $\#S = 1$ .
- $(z_1 - z_{k+1}, \dots, z_i - z_{k+1}, z_{i+1}, \dots, z_{k+1}) = (0, \dots, 0, \infty)$  for any  $0 \leq i \leq k$ , if  $k \in \mathbb{Z}_+$ ,  $l = 1$ ,  $\#S = 1$ .
- $(z_1, \dots, z_k, z_{k+1} - z_{k+2}, z_{k+2}) = (0, \dots, 0, \infty)$ , if  $k \in \mathbb{N}$ ,  $l = 2$ ,  $\#S = 1$ .
- $(z_1 - z_{k+2}, \dots, z_k - z_{k+2}, z_{k+1}, z_{k+2}) = (0, \dots, 0, \infty)$ , if  $k \in \mathbb{N}$ ,  $l = 2$ ,  $\#S = 1$ .
- $(z_1 - z_k, \dots, z_{k-1} - z_k, z_k) = (0, \dots, 0, \infty)$ , if  $k \in \mathbb{Z}_{>1}$ ,  $l = 0$ ,  $\#S = 2$ .
- $(z_1 - z_{k+1}, \dots, z_k - z_{k+1}, z_{k+1}) = (0, \dots, 0, \infty)$ , if  $k \in \mathbb{Z}_+$ ,  $l = 1$ ,  $\#S = 2$ .
- $(z_1 - z_2, z_2, z_3) = (0, \infty, 0)$ , if  $k = 0$ ,  $l = 3$ ,  $\#S = 1$ .
- $(z_1 - z_2, z_2) = (0, \infty)$ , if  $k = 0$ ,  $l = \#S = 2$ .
- $(z_1, \dots, z_{k+l})U_1A - \beta = \delta$ , for any singularities  $(z_1, \dots, z_{k+l})A - \beta = \delta$  of  $g$  listed in the above bullet points, and  $U_1 \in \mathcal{G}_{1,k,l}$ .

(c) If  $\#S > 1$ , for any  $m \in \{0, \dots, k+l-1\}$ , integers  $1 \leq i_1 < \dots < i_m \leq k+l$ , and  $(\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}) \in F_m(\mathbb{C}-S)$ , consider the multivalued analytic function with  $n-m$  variables

$$\prod_{1 \leq i < j \leq k} (z_i - z_j) \cdot f(z_1, \dots, z_n) \Big|_{\substack{z_{i_j} = \tilde{z}_{i_j} \\ j=1, \dots, m}}$$

This function satisfies the part (b) for the triple  $(k', l', S')$ , where  $k' = k - \#\{j | i_j \leq k\}$ ,  $l' = l - \#\{j | i_j > k\}$ , and  $S' = S \cup \{\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}\}$ .

(d) For the case  $\#S = 1$ , for any  $m \in \{0, \dots, k+l-1\}$ , integers  $1 \leq i_1 < \dots < i_m \leq k+l$ ,  $\psi \in \mathcal{S}_{k,l}$ , and  $(\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}) \in F_m(\mathbb{C}-S)$ , consider the multivalued analytic function with  $n-m$  variables

$$\prod_{1 \leq i < j \leq k} (z_i - z_j) \cdot f(\psi(z_1, \dots, z_n)) \Big|_{\substack{z_{i_j} = \tilde{z}_{i_j} \\ j=1, \dots, m}} \quad (3.10)$$

This function satisfies the part (b) for the triple  $(k', l', S')$ , where  $k' = k - \#\{j | i_j \leq k\}$ ,  $l' = l - \#\{j | i_j > k\}$ , and  $S' = S \cup \{\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}\}$ .

(e) This is a condition for the case  $\#S = 1$ ,  $l \geq 1$ , which is defined in the following inductive way:

We first define it for  $(k, 1, \{0\})$ ,  $k \in \mathbb{N}$ . We know, by (d), that  $z_i = 0$ ,  $z_i = \infty$ , and  $z_i - z_j = 0$  for  $i, j = 1, \dots, k+1$  with  $i \neq j$  are regular singularities of (3.10) with  $m = 0$  and  $\psi \in \mathcal{S}_{k,l}$  (See Remark 3.0.12). The coefficients of the expansions corresponding to the regular singularities listed above, as functions in  $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n$ , satisfy part (d) for either  $(k, 0, \{0\})$  if  $i = l$ , or  $(k-1, 0, \{0\})$  if  $i \neq l$ .

Let  $l \in \mathbb{Z}_{>1}$ . Suppose that this condition is defined for  $(k, l', \{0\})$  for any  $k \in \mathbb{N}$  and  $l' < l$ . Now we define it for  $(k, l, \{0\})$ . For any  $i = k+1, \dots, k+l$  and  $j = 1, \dots, k+l$  with  $i \neq j$ , we know, by (d), that  $z_i = 0$ ,  $z_i = \infty$ , and  $z_i - z_j = 0$  are regular singularities of (3.10) with  $m = 0$  and  $\psi \in \mathcal{S}_{k,l}$  (See Remark 3.0.12). The coefficients of the expansions at the singularities listed above, as functions in

$z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+l}$ , satisfy parts (d), and (e) for  $(k, l-1, \{0\})$ .

**Remark 3.0.14.** For any triple  $(k, l, S)$ , Condition 3.0.13 for  $(k, l, S)$  requires that some component-isolated singularities of the function  $g$  (part (b)), all its suitable ways of substitution and composition (part (c)) (3.10), and coefficients of the expansion at some regular singularities as a function in fewer variables (part (e)) are regular.

**Remark 3.0.15.** The part (b) of Condition 3.0.13 is an empty condition if  $\#S < l$  and  $l > 3$ .

**Remark 3.0.16.** In this work, we only interested in Condition 3.0.13 for  $(k, 1, \{0\})$ ,  $(k, 0, \{0, z\})$ , and  $(k, 2, \{0\})$ , where  $k \in \mathbb{N}$ , and  $z \in \mathbb{C}^\times$ . These three cases occur in the notion of twisted intertwining operator, the notion of  $P(z)$ -compatibility, and the convergence and extension assumption, respectively.

**Remark 3.0.17.** This remark is so far not rigorous. Non-rigorously, one can understand multivalued functions satisfying Condition 3.0.13 for  $(k, l, S)$  as functions satisfying properties that should be satisfied by the correlation function obtained from the series

$$\prod_{1 \leq i < j \leq k} (z_i - z_j)^M \cdot \langle w', Y(u_1, z_1) \cdots Y(u_k, z_k) \mathcal{Y}_1(w_1, z_{k+1}) \cdots \mathcal{Y}_l(w_l, z_{k+l}) \mathcal{Y}_{l+1}(w_{l+1}, s_1) \cdots \mathcal{Y}_{l+\#S-1}(w_{l+\#S-1}, s_{\#S-1}) w_{\#S} \rangle, \quad (3.11)$$

where  $S = \{0, s_1, \dots, s_{\#S-1}\}$ . Here, one should regard

$$\mathcal{Y}_{l+1}(w_{l+1}, s_1) \cdots \mathcal{Y}_{l+\#S-1}(w_{l+\#S-1}, s_{\#S-1}) w_{\#S}$$

as a single fixed element in  $\overline{W}$  for some twisted module  $W$ .

Define  $F_{k,l}^0(\mathbb{C} - S)$  to be the topological space which set-theoretically is equal to  $F_n(\mathbb{C} - S)$ , but the topology is given by cutting  $F_n(\mathbb{C} - S)$  along the lines  $z_i - s \in \mathbb{R}_+$ ,

with  $i = 1, \dots, n$ ,  $s \in S$ , the lines  $z_i - z_j \in \mathbb{R}_+$ , with  $1 \leq i < j \leq n$  and  $j > k$ . These lines are attached to the upper half  $(z_i - s)$ -, and  $(z_i - z_j)$ -planes. Equip  $F_{k,l}^0(\mathbb{C} - S)$  with the unique structure of complex manifold with boundary such that the set-theoretically identity map  $F_{k,l}^0(\mathbb{C} - S) \rightarrow F_n(\mathbb{C} - S)$  is holomorphic.

Define  $F_{k,l}^1(\mathbb{C} - S)$  to be the topological space which set-theoretically is equal to  $F_{k,l}(\mathbb{C} - S)$ , but the topology is given by cutting  $F_n(\mathbb{C} - S)$  along the lines  $z_i - s \in \mathbb{R}_+$ , with  $i = 1, \dots, n$ ,  $s \in S$ , the lines  $z_i - z_j \in \mathbb{R}_+$ , with  $1 \leq i < j \leq n$  and  $j > k$ . These lines are attached to the upper half  $(z_i - s)$ -, and  $(z_i - z_j)$ -planes. Equip  $F_{k,l}^0(\mathbb{C} - S)$  with the unique structure of complex manifold with boundary such that the set-theoretically identity map  $F_{k,l}^1(\mathbb{C} - S) \rightarrow F_{k,l}(\mathbb{C} - S)$  is holomorphic.

**Remark 3.0.18.** For any multi-valued analytic function  $f(z_1, \dots, z_n)$  on  $F_n(\mathbb{C} - S)$  satisfying Condition 3.0.13 for  $(k, l, S)$ , we know that any single valued branch of  $f(z_1, \dots, z_n)$  defined on a connected open subset of  $F_{k,l}^0(\mathbb{C} - S)$  can be maximally extended to single valued analytic function on  $F_{k,l}^0(\mathbb{C} - S)$ . Since  $\prod_{1 \leq i < j \leq k} (z_i - z_j)^M f(z_1, \dots, z_n)$  can be analytically extended to  $F_{k,l}(\mathbb{C} - S)$ , we also know that any single valued branch of  $\prod_{1 \leq i < j \leq k} (z_i - z_j)^M f(z_1, \dots, z_n)$  defined on a connected open subset of  $F_{k,l}^1(\mathbb{C} - S)$  can be maximally extended to single valued analytic function on  $F_{k,l}^1(\mathbb{C} - S)$ .

**Remark 3.0.19.** Recall Definition 3.0.4, Remarks 3.0.5 and 3.0.6. We can see that the space of multi-valued analytic functions on  $F_n(\mathbb{C} - S)$  satisfying Condition 3.0.13 for  $(k, l, S)$  with a preferred branch forms a **commutative associative algebra** over  $\mathbb{C}$ . Some elements in this algebra are

$$\begin{aligned} &((z_i - s)^\alpha, e^{\alpha l_p(z_i - s)}), (\log(z_i - s), l_p(z_i - s)), \\ &((z_i - z_j)^\alpha, e^{\alpha l_p(z_i - z_j)}), (\log(z_i - z_j), l_p(z_i - z_j)), \end{aligned}$$

for  $p \in \mathbb{Z}$ ,  $s \in S$ ,  $\alpha \in \mathbb{C}$ , and  $i, j = 1, \dots, k + l$ , with  $i < j$ ;

$$\text{Single-valued holomorphic functions on } F_n(\mathbb{C} - S). \quad (3.12)$$

**Proposition 3.0.20.** *Let  $k, l \in \mathbb{N}$  with  $k + l > 0$ . Let  $S$  be a finite subset of  $\mathbb{C}$  containing 0.*

1. *If  $f(z_1, \dots, z_{k+l})$  satisfies Condition 3.0.13 for  $(k, l, S)$ , then for any  $U_1 \in \mathcal{G}_{1,k,l}$ ,  $m \in \{0, \dots, k + l - 1\}$ , integers  $1 \leq i_1 < \dots < i_m \leq k + l$ , and  $(\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}) \in F_m(\mathbb{C} - S)$ , the function  $f((z_1, \dots, z_{k+l})U_1)|_{z_{i_j}=\tilde{z}_{i_j}, j=1, \dots, m}$  also satisfies Condition 3.0.13 for  $(k', l', S')$ , where  $k' = k - \#\{j | i_j \leq k\}$ ,  $l' = l - \#\{j | i_j > k\}$ , and  $S' = S \cup \{\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}\}$ .*
2. *If  $f(z_1, \dots, z_{k+l})$  satisfies Condition 3.0.13 for  $(k, l, \{0\})$ , then for any  $\psi \in \mathcal{S}_{k,l}$ ,  $m \in \{0, \dots, k + l - 1\}$ , integers  $1 \leq i_1 < \dots < i_m \leq k + l$ , and  $(\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}) \in F_m(\mathbb{C} - S)$ , the function  $f(\psi(z_1, \dots, z_{k+l}))|_{z_{i_j}=\tilde{z}_{i_j}, j=1, \dots, m}$  also satisfies Condition 3.0.13 for  $(k', l', S')$ , where  $k' = k - \#\{j | i_j \leq k\}$ ,  $l' = l - \#\{j | i_j > k\}$ , and  $S' = S \cup \{\tilde{z}_{i_1}, \dots, \tilde{z}_{i_m}\}$ . In particular,*

- (a)  *$f(z_1 - z_i, \dots, z_{i-1} - z_i, -z_i, z_{i+1} - z_i, \dots, z_{k+l} - z_i)$  for  $i = k + 1, \dots, k + l$  satisfies Condition 3.0.13 for  $(k, l, \{0\})$  when  $l \neq 0$ ;*
- (b)  *$f(z_1^{-1}, \dots, z_{k+l}^{-1})$  satisfies Condition 3.0.13 for  $(k, l, \{0\})$ .*

*Proof.* This is straightforward from Condition 3.0.13. □

**Proposition 3.0.21.** *Let  $f(z_1, \dots, z_k)$  be a multivalued function satisfies parts (a) and (b) of Condition 3.0.13 for  $(k, 0, \{0, z\})$ , where  $k \in \mathbb{Z}_+$ ,  $z \in \mathbb{C}^\times$ . Then, the function*

$$f\left(\frac{z_1 z}{z_{k+1}}, \dots, \frac{z_k z}{z_{k+1}}\right)$$

*satisfies parts (a) and (b) of Condition 3.0.13 for  $(k, 1, \{0\})$ .*

*Proof.* Notice that

$$\prod_{1 \leq i < j \leq k} \left( \frac{z_i z}{z_{k+1}} - \frac{z_j z}{z_{k+1}} \right)^M = \left( \frac{z}{z_{k+1}} \right)^{M \binom{k}{2}} \prod_{1 \leq i < j \leq k} (z_i - z_j)^M.$$

It can be checked directly that part (a) holds for  $f\left(\frac{z_1 z}{z_{k+1}}, \dots, \frac{z_k z}{z_{k+1}}\right)$ . Now we prove part (b).

1.  $(z_1, \dots, z_{k+1}) = (0, \dots, 0, \infty)$ ;

The regularity of this singularity follows from the regularity of  $(z_1, \dots, z_k) = (0, \dots, 0)$  of  $\prod_{1 \leq i < j \leq k} (z_i - z_j)^M f(z_1, \dots, z_k)$ .

2.  $(z_1, \dots, z_{k+1}) = (\infty, \dots, \infty, 0)$ ;

The regularity of this singularity follows from the regularity of  $(z_1, \dots, z_k) = (\infty, \dots, \infty)$  of  $\prod_{1 \leq i < j \leq k} (z_i - z_j)^M f(z_1, \dots, z_k)$ .

3.  $(z_1 - z_{k+1}, \dots, z_i - z_{k+1}, z_{i+1}, \dots, z_{k+1}) = (0, \dots, 0, \infty)$  for any  $0 \leq i \leq k$ ;

It follows from  $\left(\frac{z_1 z}{z_{k+1}} - z, \dots, \frac{z_i z}{z_{k+1}} - z, \frac{z_{i+1} z}{z_{k+1}}, \dots, \frac{z_k z}{z_{k+1}}\right) = (0, \dots, 0)$  being a regular singularity of  $\prod_{1 \leq i < j \leq k} (z_i - z_j)^M f\left(\frac{z_1 z}{z_{k+1}}, \dots, \frac{z_k z}{z_{k+1}}\right)$ , as  $f$  satisfies Condition 3.0.13 for  $(k, 0, \{0, z\})$ .

4. singularities obtained from those listed above by applying the permutation  $z_i \mapsto z_{\sigma(i)}$  for any  $\sigma \in \text{Sym}(k+1)$  with  $\sigma(k+1) = k+1$ .

If a singularity is obtained by applying  $\sigma$  to another singularity listed above, then use the same  $\sigma$  to modify the proof given above for the original singularity before applying  $\sigma$ .

□

**Remark 3.0.22.** *Since Condition 3.0.13 for  $(k, l, \{0\})$  is a very strong condition, and Condition 3.0.13 for  $(k, l, S)$  with  $\#S > 1$  is a much weaker condition, in general one cannot prove  $f\left(\frac{z_1 z}{z_{k+1}}, \dots, \frac{z_k z}{z_{k+1}}\right)$  satisfies the whole Condition 3.0.13 for  $(k, 1, \{0\})$  for any function  $f$  satisfying Condition 3.0.13 for  $(k, 0, \{0, z\})$ .*

### 3.0.2 Twisted intertwining operators

Now, we are ready to introduce the notion of twisted intertwining operator. Suppose  $W_1, W_2, W_3$  are three vector spaces, and

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\rightarrow W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^K \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k \end{aligned}$$

is a linear map. For  $p \in \mathbb{Z}$ , we define and denote *the  $p$ -th branch of  $\mathcal{Y}$*  as the following:

$$\begin{aligned} \mathcal{Y}^p(w_1, x)w_2 &= \sum_{k=0}^K \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 e^{2\pi i(-n-1)p} x^{-n-1}(\log x + 2p\pi i)^k \\ &= \sum_{l=0}^K \sum_{n \in \mathbb{C}} \left( \sum_{k=l}^K \binom{k}{l} (2p\pi i)^{k-l} e^{2\pi i(-n-1)p} \mathcal{Y}_{n,k}(w_1)w_2 \right) x^{-n-1}(\log x)^l \in W_3\{x\}[\log x], \end{aligned}$$

for  $w_1 \in W_1, w_2 \in W_2$ .

**Definition 3.0.23.** Let  $W_i$  be a  $g_i$ -twisted  $V$ -module,  $i = 1, 2, 3$ . A twisted intertwining operator of type  $\binom{W_3}{W_1 W_2}$  is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\rightarrow W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{k=0}^K \sum_{n \in \mathbb{C}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1}(\log x)^k \end{aligned}$$

satisfying the following conditions:

1. The *duality property*: For  $u \in V, w_1 \in W_1, w_2 \in W_2$  and  $w'_3 \in W'_3$ , there exists a multivalued analytic function  $f_{\mathcal{Y},2}(z_1, z_2; u, w_1, w_2, w'_3)$  on  $F_2(\mathbb{C} - \{0\})$  with a preferred maximally extended single-valued branch  $f_{\mathcal{Y},2}^e(z_1, z_2; u, w_1, w_2, w'_3)$  in the region given by cutting lines  $z_1 \in \mathbb{R}_+, z_2 \in \mathbb{R}_+$  and  $z_1 - z_2 \in \mathbb{R}_+$  in

$F_2(\mathbb{C} - \{0\})$ , such that the series

$$\langle w'_3, (Y_{W_3}^{g_3})^0(u, z_1) \mathcal{Y}^0(w_1, z_2) w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w'_3, (Y_{W_3}^{g_3})^0(u, z_1) \pi_n \mathcal{Y}^0(w_1, z_2) w_2 \rangle, \quad (3.13)$$

$$\langle w'_3, \mathcal{Y}^0(w_1, z_2) (Y_{W_2}^{g_2})^0(u, z_1) w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w'_3, \mathcal{Y}^0(w_1, z_2) \pi_n (Y_{W_2}^{g_2})^0(u, z_1) w_2 \rangle, \quad (3.14)$$

$$\begin{aligned} \langle w'_3, \mathcal{Y}^0((Y_{W_1}^{g_1})^0(u, z_1 - z_2) w_1, z_2) w_2 \rangle \\ = \sum_{n \in \mathbb{C}} \langle w'_3, \mathcal{Y}^0(\pi_n (Y_{W_1}^{g_1})^0(u, z_1 - z_2) w_1, z_2) w_2 \rangle \end{aligned} \quad (3.15)$$

are absolutely convergent in the regions  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$ ,  $|z_2| > |z_1 - z_2| > 0$ , respectively. Moreover, their sums are equal to  $f_{\mathcal{Y},2}^e(z_1, z_2; u, w_1, w_2, w'_3)$  in the region given by  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ , the region given by  $|z_2| > |z_1| > 0$  and  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ , the region given by  $|z_2| > |z_1 - z_2| > 0$  and  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ , respectively.

2. The *convergence for products of more than two operators*: For  $n \in \mathbb{Z}_+$ ,  $u_1, \dots, u_{n-1} \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $w'_3 \in W'_3$ , the series

$$\begin{aligned} \langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) \mathcal{Y}(w_1, z_n) w_2 \rangle \\ = \sum_{m_1, \dots, m_{n-1} \in \mathbb{C}} \langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \pi_{m_1} \cdots \pi_{m_{n-2}} Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) \pi_{m_{n-1}} \mathcal{Y}(w_1, z_n) w_2 \rangle \end{aligned}$$

is absolutely convergent on the region  $|z_1| > \cdots > |z_n| > 0$  and can be maximally extended to a multivalued analytic function

$$f_{\mathcal{Y},n}(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3)$$

on the region  $F_n(\mathbb{C} - \{0\})$  satisfying the Condition 3.0.13 for  $(n - 1, 1, \{0\})$ .

3. The  $L(-1)$ -derivative property: for any  $w_1 \in W_1$ ,

$$\frac{d}{dx} \mathcal{Y}(w_1, x) = \mathcal{Y}(L(-1)w_1, x).$$

In the rest of this paper, we assume that  $\mathcal{Y}$  is a twisted intertwining operator of type  $\binom{W_3}{W_1 W_2}$ .

**Remark 3.0.24.** Just like the preferred branch  $f_{\mathcal{Y},2}^e$ , now we define the preferred branch  $f_{\mathcal{Y},n}^e$  of the multi-valued function  $f_{\mathcal{Y},n}$  for  $n \in \mathbb{Z}_+$ . Let

$$f_{\mathcal{Y},n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3)$$

be a single valued analytic function on  $F_{n-1,1}^0(\mathbb{C} - \{0\})$  such that

$$\begin{aligned} f_{\mathcal{Y},n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3) = \\ \langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) \mathcal{Y}(w_1, z_n) w_2 \rangle \end{aligned}$$

on the region given by  $|z_1| > \cdots > |z_n| > 0$ , and  $|\arg(z_i - z_n) - \arg z_i| < \frac{\pi}{2}$ ,  $i = 1, \dots, n-1$ .

**Definition 3.0.25.** For a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\rightarrow W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} \mathcal{Y}_{n,k}(w_1)w_2 x^{-n-1} \log x^k, \end{aligned} \quad (3.16)$$

we say  $\mathcal{Y}$  satisfies the *full lower truncation property* or is *fully lower truncated*, if

For any  $n \in \mathbb{C}$ ,  $k \in \mathbb{N}$ , and  $w_1 \in W_1, w_2 \in W_2, w'_3 \in W'_3$ ,

$$\begin{aligned} \mathcal{Y}_{n+l,k}(w_1)(w_2) = 0, \quad \langle w'_3, \mathcal{Y}_{n-l,k}(w_1) \cdot \rangle = 0, \quad \langle w'_3, \mathcal{Y}_{n-l,k}(\cdot)w_2 \rangle = 0, \\ \text{for sufficiently large } l \in \mathbb{N}. \end{aligned} \quad (3.17)$$

If only the condition that  $\mathcal{Y}_{n+l,k}(w_1)(w_2) = 0$  for sufficiently large  $l \in \mathbb{N}$  is satisfied, we say  $\mathcal{Y}$  satisfies the *lower truncation property* or is *lower truncated*. We say a module  $(W, Y_W)$  is *fully lower truncated* (or *lower truncated*) if  $Y_W$  is fully truncated (or lower truncated).

**Remark 3.0.26.** *Similar as Lemma 2.0.4, we do not require lower truncation property in Definition 3.0.23. However, it automatically holds when modules are lower-bounded.*

*Suppose  $\mathcal{Y}$  is a twisted intertwining operator of type  $\binom{W_3}{W_1 W_2}$ . If  $W_3$  is lower-bounded, then the first condition in the full truncation property, i.e. for any fixed  $n, k, w_1, w_2$ ,  $\mathcal{Y}_{n+l,k}(w_1)(w_2) = 0$  for sufficiently large  $l \in \mathbb{N}$ , automatically holds. If  $W_2$  or  $W_1$  is lower-bounded, the second or third condition in the full truncation property automatically holds, respectively. Therefore, twisted intertwining operator among three lower-bounded twisted modules is automatically fully truncated.*

*On the other hand, without lower-boundedness or other properties, truncation property cannot be derived only from the definition of twisted module.*

We can generalize the lemma 7.7 and proposition 7.9 in [HLZ5] to the following two lemmas respectively.

**Lemma 3.0.27.** *Let  $b_1, b_2 \in \mathbb{R}$ ,  $b_1 < b_2$ ,  $p \in \mathbb{Z}$ , and  $D$  is a subset of  $\{z \in \mathbb{C} | b_1 \leq \Im z \leq b_2\}$ . Suppose a series  $\sum_{\alpha \in D} a_\alpha e^{\alpha p(z)}$  ( $a_\alpha \in \mathbb{C}$ ) is absolutely convergent on an open subset  $\Omega$  of  $\mathbb{C}^\times$ . Then the series*

$$\sum_{\alpha \in D} a_\alpha \alpha e^{\alpha p(z)}$$

*is absolutely and uniformly convergent near any  $z \in \Omega$ . In particular,  $\sum_{\alpha \in D} a_\alpha e^{\alpha p(z)}$  is holomorphic on  $\Omega$ .*

*Proof.* Let  $B = \max\{|b_1|, |b_2|\}$ ,  $M = e^{2\pi B(1+|p|)}$ . For any  $z \in \mathbb{C}^\times, \alpha \in D$ , since

$|e^{\alpha l_p(z)}| = |z|^{\Re \alpha} e^{-\Im \alpha \arg z} e^{-2\pi p \Im \alpha}$ , we have

$$M^{-1}|z|^{\Re \alpha} < |e^{\alpha l_p(z)}| < M|z|^{\Re \alpha},$$

or equivalently,  $M^{-1}|e^{\alpha l_p(z)}| < |z|^{\Re \alpha} < M|e^{\alpha l_p(z)}|$ .

We know that  $\sum_{\Re \alpha \geq 0} a_\alpha e^{\alpha l_p(z)}$  and  $\sum_{\Re \alpha < 0} a_\alpha e^{\alpha l_p(z)}$  are all absolutely convergent in  $\Omega$ . For any  $\alpha \in D$ , we have  $b_1 \leq \Im \alpha \leq b_2$ . Thus,  $|\alpha| \leq |\Re \alpha| + B$ . Fix  $z_0 \in \Omega$ , there exists  $z_1, z_2 \in \Omega$ ,  $r_1, r_2 > 0$ , such that  $|z_1| < r_1 < |z_0| < r_2 < |z_2|$ . Since

$$\sqrt[\Re \alpha]{|\alpha|} < \sqrt[\Re \alpha]{\Re \alpha + B} \rightarrow 1 \quad (\text{when } \Re \alpha \rightarrow +\infty),$$

there exists  $L > 0$  such that  $\sqrt[\Re \alpha]{|\alpha|} < \min \left\{ \frac{r_1}{|z_1|}, \frac{|z_2|}{r_2} \right\}$  when  $\Re \alpha > L$ . When  $z \in \Omega$ ,  $r_1 < |z| < r_2$ , and  $\Re \alpha > L$ , we have  $\sqrt[\Re \alpha]{|\alpha|} < \min \left\{ \frac{r_1}{|z_1|}, \frac{|z_2|}{r_2} \right\} < \min \left\{ \frac{|z|}{|z_1|}, \frac{|z_2|}{|z|} \right\}$ . So, for  $z \in \Omega$ ,  $r_1 < |z| < r_2$ , we have

$$\begin{aligned} \sum_{\substack{\alpha \in D \\ \Re \alpha > L}} |a_\alpha \alpha e^{\alpha l_p(z)}| &< M \sum_{\substack{\alpha \in D \\ \Re \alpha > L}} |a_\alpha \alpha| |z|^{\Re \alpha} \\ &< M \sum_{\substack{\alpha \in D \\ \Re \alpha > L}} |a_\alpha| \left( \sqrt[\Re \alpha]{|\alpha|} |z| \right)^{\Re \alpha} < M \sum_{\substack{\alpha \in D \\ \Re \alpha > L}} |a_\alpha| \left( \frac{|z_2|}{r_2} |z| \right)^{\Re \alpha} \\ &< M \sum_{\substack{\alpha \in D \\ \Re \alpha > L}} |a_\alpha| \cdot |z_2|^{\Re \alpha} < M^2 \sum_{\substack{\alpha \in D \\ \Re \alpha > L}} |a_\alpha e^{\alpha l_p(z_2)}| < \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{\substack{\alpha \in D \\ \Re \alpha < -L}} |a_\alpha \alpha e^{\alpha l_p(z)}| &< M \sum_{\substack{\alpha \in D \\ \Re \alpha < -L}} |a_\alpha \alpha| |1/z|^{-\Re \alpha} \\ &< M \sum_{\substack{\alpha \in D \\ \Re \alpha < -L}} |a_\alpha| \left( \sqrt[-\Re \alpha]{|\alpha|} / |z| \right)^{-\Re \alpha} < M \sum_{\substack{\alpha \in D \\ \Re \alpha < -L}} |a_\alpha| |1/z_1|^{-\Re \alpha} \\ &< M^2 \sum_{\substack{\alpha \in D \\ \Re \alpha < -L}} |a_\alpha e^{\alpha l_p(z_1)}| < \infty. \end{aligned}$$

Also,

$$\begin{aligned}
\sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} |a_\alpha \alpha e^{\alpha l_p(z)}| &< \sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} (|\Re \alpha| + B) |a_\alpha e^{\alpha l_p(z)}| \leq (L + B) \sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} |a_\alpha e^{\alpha l_p(z)}| \\
&= (L + B) \sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} |a_\alpha| |z|^{\Re \alpha} e^{-\Im \alpha \arg z} e^{-2\pi p \Im \alpha} \\
&< (L + B) e^{2\pi B} \sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} |a_\alpha| |z_2|^{\Re \alpha} e^{-2\pi p \Im \alpha} \\
&< (L + B) e^{4\pi B} \sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} |a_\alpha| |z_2|^{\Re \alpha} e^{-\Im \alpha \arg z_2} e^{-2\pi p \Im \alpha} \\
&= (L + B) e^{4\pi B} \sum_{\substack{\alpha \in D \\ 0 \leq \Re \alpha \leq L}} |a_\alpha e^{\alpha l_p(z_2)}| < \infty.
\end{aligned}$$

Similarly,

$$\sum_{\substack{\alpha \in D \\ -L \leq \Re \alpha < 0}} |a_\alpha \alpha e^{\alpha l_p(z)}| < (L + B) e^{4\pi B} \sum_{\substack{\alpha \in D \\ -L \leq \Re \alpha < 0}} |a_\alpha e^{\alpha \cdot l_p(z_1)}| < \infty.$$

Therefore,  $\sum_{\alpha \in D} a_\alpha \alpha e^{\alpha l_p(z)}$  is absolutely and uniformly convergent near  $z$ . Here, uniformly convergence near  $z$  means that there exists an open neighborhood of  $z$  such that the series is uniformly convergent on this open neighborhood.  $\square$

**Lemma 3.0.28.** *Let  $b_1, b_2 \in \mathbb{R}$ ,  $b_1 < b_2$ ,  $N$  be a nonnegative integer,  $\Omega$  be a nonempty open subset of  $\mathbb{C}^\times$ , and  $D$  be a subset of  $\{z \in \mathbb{C} | b_1 \leq \Im z \leq b_2\}$ . Let  $a_{\alpha, \beta} \in \mathbb{C}$  for  $\alpha \in D$ ,  $\beta = 0, \dots, N$ . Then the series*

$$\sum_{\alpha \in D} a_{\alpha, \beta} z^\alpha \quad (z^\alpha \text{ means } e^{\alpha \cdot l_0(z)}) \tag{3.18}$$

for  $\beta = 0, \dots, N$  are all absolutely convergent on  $\Omega$  if and only if the series

$$\sum_{\alpha \in D} \left( \frac{\partial}{\partial z} \right)^k \left( \left( \sum_{\beta=0}^N a_{\alpha, \beta} \log(z)^\beta \right) z^\alpha \right) \quad (3.19)$$

for  $k \in \mathbb{N}$  are all absolutely convergent on  $\Omega$ .

*Proof.* Let  $B = 2 \max\{|b_1|, |b_2|\}$ , and  $M = e^{2\pi B}$ . For any  $\alpha \in D$ ,  $z \in \mathbb{C}^\times$ , we have

$$M^{-1}|z|^{\Re \alpha} \leq |z^\alpha| = |z|^{\Re \alpha} e^{-\Im \alpha \cdot \arg z} < M|z|^{\Re \alpha}$$

For any  $z, z' \in \mathbb{C}^\times$ ,  $\alpha \in D$ , with  $|z| < |z'|$ , and  $\Re \alpha \geq 0$ , we have

$$|z^\alpha| = |z|^{\Re \alpha} e^{-\Im \alpha \cdot \arg z} \leq \left( |z'|^{\Re \alpha} e^{-\Im \alpha \cdot \arg z'} \right) e^{-\Im \alpha \cdot (\arg z - \arg z')} < M^2 |z'^\alpha|$$

Therefore, the proof of the proposition 7.9 in [HLZ5] still works if we change  $\sum_{n \geq 0}$  to  $\sum_{\alpha \in D, \Re \alpha \geq 0}$ , and put an extra  $M^2$  suitably on the larger side of each inequality.  $\square$

Therefore, we have the following result, which corresponds to Proposition 7.14 in [HLZ5].

**Lemma 3.0.29.** *The multiple series*

$$\sum_{m, n \in \mathbb{C}} \sum_{i, j=0}^K \left\langle w'_3, (Y_{W_3}^{g_3})_{m, j}(u) \mathcal{Y}_{n, i}(w_1) w_2 \right\rangle \cdot e^{(-m-1)l_0(z_1)} l_0(z_1)^j e^{(-n-1)l_0(z_2)} l_0(z_2)^i, \quad (3.20)$$

$$\sum_{m, n \in \mathbb{C}} \sum_{i, j=0}^K \left\langle w'_3, \mathcal{Y}_{n, i}(w_1) (Y_{W_2}^{g_2})_{m, j}(u) w_2 \right\rangle \cdot e^{(-m-1)l_0(z_1)} l_0(z_1)^j e^{(-n-1)l_0(z_2)} l_0(z_2)^i, \quad (3.21)$$

$$\sum_{m, n \in \mathbb{C}} \sum_{i, j=0}^K \left\langle w'_3, \mathcal{Y}_{n, i} \left( (Y_{W_1}^{g_1})_{m, j}(u) w_1 \right) w_2 \right\rangle \cdot e^{(-m-1)l_0(z_1 - z_2)} l_0(z_1 - z_2)^j e^{(-n-1)l_0(z_2)} l_0(z_2)^i \quad (3.22)$$

are absolutely convergent to their corresponding iterate sum (3.13), (3.14), (3.15), respectively, wherever (3.13), (3.14), (3.15) are absolutely convergent, respectively.

**Lemma 3.0.30.** *For any  $p_1, p_2, p_{12} \in \mathbb{Z}$ , the series*

$$\langle w'_3, (Y_{W_3}^{g_3})^{p_1}(u, z_1) \mathcal{Y}^{p_2}(w_1, z_2) w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w'_3, (Y_{W_3}^{g_3})^{p_1}(u, z_1) \pi_n \mathcal{Y}^{p_2}(w_1, z_2) w_2 \rangle, \quad (3.23)$$

$$\langle w'_3, \mathcal{Y}^{p_2}(w_1, z_2) (Y_{W_2}^{g_2})^{p_1}(u, z_1) w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w'_3, \mathcal{Y}^{p_2}(w_1, z_2) \pi_n (Y_{W_2}^{g_2})^{p_1}(u, z_1) w_2 \rangle, \quad (3.24)$$

$$\langle w'_3, \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2) w_1, z_2) w_2 \rangle = \sum_{n \in \mathbb{C}} \langle w'_3, \mathcal{Y}^{p_2}(\pi_n (Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2) w_1, z_2) w_2 \rangle \quad (3.25)$$

are absolutely convergent in the regions  $|z_1| > |z_2| > 0$ ,  $|z_2| > |z_1| > 0$ ,  $|z_2| > |z_1 - z_2| > 0$ , respectively. Moreover, their sums are equal to the branches

$$\begin{aligned} & f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3), \\ & f_{\mathcal{Y},2}^{(b_{12}^{-1}b_{13}b_{12})^{p_1}(b_{23}b_{12})^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3), \\ & f_{\mathcal{Y},2}^{(b_{23}b_{13})^{p_2}b_{12}^{p_{12}}}(z_1, z_2; u, w_1, w_2, w'_3), \end{aligned}$$

of  $f_{\mathcal{Y},2}(z_1, z_2; u, w_1, w_2, w'_3)$ , respectively, in the region given by  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ , the region given by  $|z_2| > |z_1| > 0$  and  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ , the region given by  $|z_2| > |z_1 - z_2| > 0$  and  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ , respectively.

Moreover, the multiple series

$$\sum_{m,n \in \mathbb{C}} \sum_{i,j=0}^K \left\langle w'_3, (Y_{W_3}^{g_3})_{m,j}(u) \mathcal{Y}_{n,i}(w_1) w_2 \right\rangle \cdot e^{(-m-1)l_{p_1}(z_1)} l_{p_1}(z_1)^j e^{(-n-1)l_{p_2}(z_2)} l_{p_2}(z_2)^i, \quad (3.26)$$

$$\sum_{m,n \in \mathbb{C}} \sum_{i,j=0}^K \left\langle w'_3, \mathcal{Y}_{n,i}(w_1) (Y_{W_2}^{g_2})_{m,j}(u) w_2 \right\rangle \cdot e^{(-m-1)l_{p_1}(z_1)} l_{p_1}(z_1)^j e^{(-n-1)l_{p_2}(z_2)} l_{p_2}(z_2)^i, \quad (3.27)$$

$$\sum_{m,n \in \mathbb{C}} \sum_{i,j=0}^K \left\langle w'_3, \mathcal{Y}_{n,i} \left( (Y_{W_1}^{g_1})_{m,j}(u) w_1 \right) w_2 \right\rangle \cdot e^{(-m-1)l_{p_{12}}(z_1-z_2)} l_{p_{12}}(z_1-z_2)^j e^{(-n-1)l_{p_2}(z_2)} l_{p_2}(z_2)^i \quad (3.28)$$

are absolutely convergent to their corresponding iterate sum (3.23), (3.24), (3.25), respectively, wherever (3.23), (3.24), (3.25) are absolutely convergent, respectively.

*Proof.* Without loss of generality, we can assume that  $u, w_1, w_2, w'_3$  are all homogeneous elements. Let  $\Delta = -\text{wt } w'_3 + \text{wt } u + \text{wt } w_1 + \text{wt } w_2$ , and

$$M = \max \left\{ |\Im n| \mid n \in \mathbb{C}, \langle w'_3, Y_{W_3}^{g_1 g_2}(u, x) \mathcal{Y}_{n,k}(w_1) w_2 \rangle \neq 0 \text{ for some } k = 0, \dots, K \right\}.$$

Notice that  $M$  is well-defined, since the set to which we take maximum is finite. First, we need to prove that (3.26) is absolutely convergent when  $|z_1| > |z_2| > 0$ . Notice that if the multiple sum (3.26) is absolutely convergent, then the corresponding iterate sum (3.23) is automatically absolutely convergent on the same region, and they are equal. To prove the absolute convergence of (3.26), we only need to show that for each  $i, j = 0, \dots, K$ , the right-hand side of the following inequality is convergent when  $|z_1| > |z_2| > 0$ .

$$\begin{aligned} & \sum_{m,n \in \mathbb{C}} \left| \left\langle w'_3, (Y_{W_3}^{g_3})_{m,j}(u) \mathcal{Y}_{n,i}(w_1) w_2 \right\rangle e^{(-m-1)l_{p_1}(z_1)} e^{(-n-1)l_{p_2}(z_2)} \right| \\ &= \sum_{n \in \mathbb{C}} \left| \left\langle w'_3, (Y_{W_3}^{g_3})_{\Delta-n-2,j}(u) \mathcal{Y}_{n,i}(w_1) w_2 \right\rangle e^{(-\Delta+n+1)l_0(z_1)} e^{(-n-1)l_0(z_2)} \right| \\ & \qquad \qquad \qquad \left| e^{2\pi i [(-\Delta+n+1)p_1 + (-n-1)p_2]} \right| \\ &= \sum_{n \in \mathbb{C}} e^{2\pi p_1 \Im \Delta} e^{2\pi(p_2-p_1) \Im n}. \end{aligned}$$

$$\begin{aligned}
& \left| \left\langle w'_3, (Y_{W_3}^{g_3})_{\Delta-n-2,j}(u) \mathcal{Y}_{n,i}(w_1) w_2 \right\rangle e^{(-\Delta+n+1)l_0(z_1)} e^{(-n-1)l_0(z_2)} \right| \\
& \leq e^{2\pi p_1 \Im \Delta + 2\pi |p_2 - p_1| M} . \\
& \sum_{n \in \mathbb{C}} \left| \left\langle w'_3, (Y_{W_3}^{g_3})_{\Delta-n-2,j}(u) \mathcal{Y}_{n,i}(w_1) w_2 \right\rangle e^{(-\Delta+n+1)l_0(z_1)} e^{(-n-1)l_0(z_2)} \right| \\
& = e^{2\pi p_1 \Im \Delta + 2\pi |p_2 - p_1| M} \cdot \sum_{m,n \in \mathbb{C}} \left| \left\langle w'_3, (Y_{W_3}^{g_3})_{m,j}(u) \mathcal{Y}_{n,i}(w_1) w_2 \right\rangle e^{(-m-1)l_0(z_1)} e^{(-n-1)l_0(z_2)} \right|
\end{aligned} \tag{3.29}$$

By lemma 3.0.29, the right-hand side of (3.29) is absolutely convergent on the region  $|z_1| > |z_2| > 0$ . So, the left-hand side of (3.29) is also absolutely convergent on the same region. The absolute convergence of (3.27) and (3.28) are similar.

Since  $(z_1, z_2) = (\infty, 0)$  is a regular singularity of  $f_{y,2}(z_1, z_2; u, w_1, w_2, w'_3)$ , and  $Y_{W_3}^{g_3}(u, x) \in x^{-a} \text{End}(W_3)[[x, x^{-1}]][\log(x)]$  for any  $a \in \mathbb{C}^\times$ ,  $u \in V^{[a+\mathbb{Z}]}$ , we know that there exists a finite set  $A \subset \mathbb{C}/\mathbb{Z}$ , and  $R_\mu \in \mu$  for  $\mu \in A$ , such that for any  $z_1, z_2 \in \mathbb{C}$  with  $|z_1| > |z_2| > 0$ , the series (3.20), therefore (3.13), is equal to

$$\begin{aligned}
& \sum_{\mu \in A} \sum_{l \in \mathbb{Z}_{>0}} \sum_{i,j=0}^K \left\langle w'_3, (Y_{W_3}^{g_3})_{\Delta-R_\mu+l-2,j}(u) \mathcal{Y}_{R_\mu-l,i}(w_1) w_2 \right\rangle \cdot \\
& \quad e^{(-\Delta+R_\mu-l+1)l_0(z_1)} l_0(z_1)^j e^{(-R_\mu+l-1)l_0(z_2)} l_0(z_2)^i \\
& = \sum_{\mu \in A} \sum_{i,j=0}^K e^{(-\Delta+R_\mu)l_0(z_1)} e^{-R_\mu l_0(z_2)} l_0(z_1)^j l_0(z_2)^i \cdot \\
& \quad \sum_{l \in \mathbb{Z}_{>0}} \left\langle w'_3, (Y_{W_3}^{g_3})_{\Delta-R_\mu+l-2,j}(u) \mathcal{Y}_{R_\mu-l,i}(w_1) w_2 \right\rangle \left( \frac{z_2}{z_1} \right)^{l-1} \\
& = \sum_{\mu \in A} \sum_{i,j=0}^K e^{(-\Delta+R_\mu)l_0(z_1)} e^{-R_\mu l_0(z_2)} l_0(z_1)^j l_0(z_2)^i \cdot f_{\mu;i,j} \left( \frac{z_2}{z_1} \right), \tag{3.30} \\
& \quad \text{where } f_{\mu;i,j}(z) = \sum_{l \in \mathbb{Z}_{>0}} \left\langle w'_3, (Y_{W_3}^{g_3})_{\Delta-R_\mu+l-2,j}(u) \mathcal{Y}_{R_\mu-l,i}(w_1) w_2 \right\rangle z^{l-1}.
\end{aligned}$$

By the duality of  $\mathcal{Y}$  and Lemma 3.0.29, we know that for each  $\mu \in A$ , and  $i, j = 0, \dots, K$ , the radius of convergence of the power series  $f_{\mu;i,j}$  is larger than

or equal to 1. Since  $f_{\mu;i,j}$  is single valued, from (3.30) we can see that applying  $(b_{13}b_{12})^{p_1}b_{23}^{p_2}$  to the right-hand side of (3.30), one gets

$$\begin{aligned}
& \left( (b_{13}b_{12})^{p_1}b_{23}^{p_2} \right) \cdot \left[ \sum_{\mu \in A} \sum_{i,j=0}^K e^{(-\Delta+R_\mu)l_0(z_1)} e^{-R_\mu l_0(z_2)} l_0(z_1)^j l_0(z_2)^i f_{\mu;i,j} \left( \frac{z_2}{z_1} \right) \right] \\
&= \sum_{\mu \in A} \sum_{i,j=0}^K \left( (b_{13}b_{12})^{p_1}b_{23}^{p_2} \right) \cdot \left[ e^{(-\Delta+R_\mu)l_0(z_1)} e^{-R_\mu l_0(z_2)} l_0(z_1)^j l_0(z_2)^i \right] f_{\mu;i,j} \left( \frac{z_2}{z_1} \right) \\
&= \sum_{\mu \in A} \sum_{i,j=0}^K e^{(-\Delta+R_\mu)l_{p_1}(z_1)} e^{-R_\mu l_{p_2}(z_2)} l_{p_1}(z_1)^j l_{p_2}(z_2)^i f_{\mu;i,j} \left( \frac{z_2}{z_1} \right) \\
&= \sum_{\mu \in A} \sum_{l \in \mathbb{Z}_{>0}} \sum_{i,j=0}^K \left\langle w'_3, \left( Y_{W_3}^{g_3} \right)_{\Delta-R_\mu-l-2,j} (u) \mathcal{Y}_{R_\mu+l,i} (w_1) w_2 \right\rangle \cdot \\
&\quad e^{(-\Delta+R_\mu+l+1)l_{p_1}(z_1)} l_{p_1}(z_1)^j e^{(-R_\mu-l-1)l_{p_2}(z_2)} l_{p_2}(z_2)^i \\
&= \text{the multiple series (3.26)}. \tag{3.31}
\end{aligned}$$

On the other hand, by the duality of  $\mathcal{Y}$ , in the region given by  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ , applying  $(b_{13}b_{12})^{p_1}b_{23}^{p_2}$  to the left-hand side of (3.30) (which is just (3.13)), one gets

$$f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3). \tag{3.32}$$

Thus, (3.26), and therefore (3.23), is absolutely convergent to (3.32) in the region given by  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ .

For the other two series, the proofs are similar.  $\square$

We define the following group elements, for  $p_1, p_2, p_{12} \in \mathbb{Z}$ ,

$$\begin{aligned}
a^{p_1,p_2} &= (b_{13}b_{12})^{p_1} b_{23}^{p_2}, \\
b^{p_1,p_2} &= (b_{12}^{-1}b_{13}b_{12})^{p_1} (b_{23}b_{12})^{p_2}, \\
c^{p_2,p_{12}} &= (b_{23}b_{13})^{p_2} b_{12}^{p_{12}}.
\end{aligned}$$

**Corollary 3.0.31.** *For any twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1W_2}$  and integer  $q \in \mathbb{Z}$ , the  $q$ -th branch  $\mathcal{Y}^q$  of  $\mathcal{Y}$  is a twisted intertwining operator of type  $\binom{\phi_{g_3^q}(W_3)}{\phi_{g_1^q}(W_1) \phi_{g_2^q}(W_2)}$ . Therefore, when  $W_i$ ,  $i=1,2,3$  are  $g_i$ -twisted modules with  $g_i$ -action respectively, by proposition 2.0.6, the map  $w_1 \otimes w_2 \mapsto g_3^{-q}\mathcal{Y}^q(g_1^{-q}w_1, x)g_2^{-q}w_2$  is a twisted intertwining operator of type  $\binom{W_3}{W_1W_2}$ .*

**Remark 3.0.32.** *We also denote  $\mathcal{Y}^q(u, x)$  by  $\mathcal{Y}(u, e^{2\pi i q}x)$ , and denote*

$\mathcal{Y}(w, y)|_{y^n=e^{2\pi i q n}x^n}, n \in \mathbb{C}$  *by*  $\mathcal{Y}(w, y)|_{y=e^{2\pi i q}x}$ .

*Proof.* Notice that

$$\begin{aligned}
& \left\langle w'_3, (\phi_{g_3^q}(Y_{W_3}^{g_3}))^0(u, z_1) (\mathcal{Y}^q)^0(w_1, z_2)w_2 \right\rangle \\
&= \left\langle w'_3, (Y_{W_3}^{g_3})^0(g_3^{-q}u, z_1)\mathcal{Y}^q(w_1, z_2)w_2 \right\rangle \\
&= \left\langle w'_3, (Y_{W_3}^{g_3})^q(u, z_1)\mathcal{Y}^q(w_1, z_2)w_2 \right\rangle, \\
& \left\langle w'_3, (\mathcal{Y}^q)^0(w_1, z_2) (\phi_{g_2^q}(Y_{W_2}^{g_2}))^0(u, z_1)w_2 \right\rangle \\
&= \left\langle w'_3, \mathcal{Y}^q(w_1, z_2) (Y_{W_2}^{g_2})^0(g_2^{-q}u, z_1)w_2 \right\rangle \\
&= \left\langle w'_3, \mathcal{Y}^q(w_1, z_2) (Y_{W_2}^{g_2})^q(u, z_1)w_2 \right\rangle, \\
& \left\langle w'_3, (\mathcal{Y}^q)^0\left((\phi_{g_1^q}(Y_{W_1}^{g_1}))^0(u, z_1 - z_2)w_1, z_2\right)w_2 \right\rangle \\
&= \left\langle w'_3, \mathcal{Y}^q\left((Y_{W_1}^{g_1})^0(g_1^{-q}u, z_1 - z_2)w_1, z_2\right)w_2 \right\rangle \\
&= \left\langle w'_3, \mathcal{Y}^q\left((Y_{W_1}^{g_1})^q(u, z_1 - z_2)w_1, z_2\right)w_2 \right\rangle.
\end{aligned}$$

Notice that  $(b_{13}b_{12})^qb_{23}^q = (b_{12}^{-1}b_{13}b_{12})^q(b_{23}b_{12})^q = (b_{23}b_{13})^qb_{12}^q = (b_{13}b_{12}b_{23})^q$ . By lemma (3.0.30), we know that the above three series are absolutely convergent on the region given by  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ , the region given by  $|z_2| > |z_1| > 0$  and  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ , the region given by  $|z_2| > |z_1 - z_2| > 0$  and  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ , respectively, to the function

$$f_{\mathcal{Y},2}^{(b_{13}b_{12}b_{23})^q}(z_1, z_2; u, w_1, w_2, w'_3),$$

which means that the duality for  $\mathcal{Y}^q$  has been proven.

For  $L(-1)$ -derivative property, we have

$$\begin{aligned} \mathcal{Y}^q(L(-1)w, x) &= \mathcal{Y}(L(-1)w, e^{2\pi i q}x) = \\ &= \left( \frac{d}{dy} \mathcal{Y}(w, y) \right) \Big|_{y=e^{2\pi i q}x} = \frac{d}{dx} \mathcal{Y}(w, e^{2\pi i q}x). \end{aligned} \quad (3.33)$$

The last step is because for any  $m \in \mathbb{C}$ , we have

$$\begin{aligned} \left( \frac{d}{dy} y^m \right) \Big|_{y=e^{2\pi i q}x} &= ((m-1)y^{m-1}) \Big|_{y=e^{2\pi i q}x} = \\ &= (m-1)e^{2\pi i(m-1)q}x^{m-1} = e^{2\pi i m q} \left( \frac{d}{dx} x^m \right). \end{aligned}$$

The part 1 and 3 in the definition of twisted intertwining operator are straightforward.  $\square$

**Lemma 3.0.33.** *For any  $p_1, p_2, p_{12} \in \mathbb{Z}$ ,*

- (a) *The series (3.23) is absolutely convergent to*

$$\begin{aligned} &f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}b_{12}}(z_1, z_2; u, w_1, w_2, w'_3), \\ &f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}b_{12}^{-1}}(z_1, z_2; u, w_1, w_2, w'_3) \end{aligned}$$

*in the region given by  $|z_1| > |z_2| > 0$  and  $-2\pi < \arg(z_1 - z_2) - \arg z_1 < -\frac{3\pi}{2}$ , and the region given by  $|z_1| > |z_2| > 0$  and  $\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < 2\pi$ , respectively.*

- (b) *The series (3.24) is absolutely convergent to*

$$f_{\mathcal{Y},2}^{(b_{12}^{-1}b_{13}b_{12})^{p_1}(b_{23}b_{12})^{p_2}b_{12}^{-1}}(z_1, z_2; u, w_1, w_2, w'_3)$$

*in the region given by  $|z_2| > |z_1| > 0$  and  $\frac{\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < \frac{3\pi}{2}$ .*

- (c) The series (3.25) is absolutely convergent to

$$f_{\mathcal{Y},2}^{(b_{23}b_{13})^{p_2}b_{12}^{p_1}b_{13}^{-1}}(z_1, z_2; u, w_1, w_2, w'_3),$$

$$f_{\mathcal{Y},2}^{(b_{23}b_{13})^{p_2}b_{12}^{p_1}b_{13}^{-1}}(z_1, z_2; u, w_1, w_2, w'_3)$$

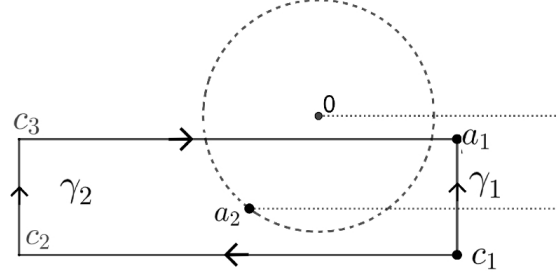
in the region given by  $|z_2| > |z_1 - z_2| > 0$  and  $-2\pi < \arg z_1 - \arg z_2 < -\frac{3\pi}{2}$ ,  
and the region given by  $|z_2| > |z_1 - z_2| > 0$  and  $\frac{3\pi}{2} < \arg z_1 - \arg z_2 < 2\pi$ ,  
respectively.

*Proof.* We only prove the first part of (a), since the second part of (a), (b), and (c) are similar.

Fix  $a_1, a_2 \in \mathbb{C}$  such that  $|a_1| > |a_2| > 0$  and  $-2\pi < \arg(a_1 - a_2) - \arg a_1 < -\frac{3\pi}{2}$ . We know that  $\Im(a_1), \Im(a_2) < 0$ . Let  $c_1 = a_1 - i|a_2|$  and  $\gamma_1$  be the path given by  $\gamma_1(t) = ((1-t)c_1 + ta_1, a_2)$ ,  $t \in [0, 1]$ . By lemma 3.0.30, the series (3.23) is convergent to  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$  near  $(c_1, a_2)$ , because  $|c_1| > |a_2| > 0$  and  $|\arg(c_1 - a_2) - \arg c_1| < \frac{\pi}{2}$ . Since the image of  $\gamma_1$  is contained in the region  $\{(z_1, z_2) \mid |z_1| > |z_2| > 0, z_1, z_2 \notin \mathbb{R}_+\}$  where the series (2.1) is holomorphic, and also (2.1) is convergent to  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$  near  $\gamma_1(0) = (c_1, a_2)$ , doing analytic continuation of  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$  along  $\gamma_1$ , one gets  $\langle w'_3, (Y_{W_3}^{g_3})^{p_1}(u, a_1)\mathcal{Y}^{p_2}(w_1, a_2)w_2 \rangle$ . Let  $c_2 = c_1 - 3|a_2|$ ,  $c_3 = c_2 + i|a_2|$ , and  $\gamma_2$  be the path given by  $\gamma_2(t) = ((1-3t)c_1 + 3tc_2, a_2)$ , if  $t \in [0, 1/3]$ ,  $((2-3t)c_2 + (3t-1)c_3, a_2)$ , if  $t \in [1/3, 2/3]$ ,  $((3-3t)c_3 + (3t-2)a_1, a_2)$ , if  $t \in [2/3, 1]$ . Since the image of  $\gamma_2$  is contained in  $\{(z_1, z_2) \mid z_1, z_2, z_1 - z_2 \notin \mathbb{R}_+\}$  where  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$  is holomorphic, doing analytic continuation of  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$  along  $\gamma_2$ , one gets  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(a_1, a_2; u, w_1, w_2, w'_3)$ .

Therefore, doing analytic continuation of  $f_{\mathcal{Y},2}^{(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$  along  $\gamma_2^{-1}\gamma_1$ , one gets  $\langle w'_3, (Y_{W_3}^{g_3})^{p_1}(u, a_1)\mathcal{Y}^{p_2}(w_1, a_2)w_2 \rangle$ . The path  $\gamma_2^{-1}\gamma_1$  means letting  $z_1$  go around  $z_2$  counterclockwise, which corresponds to the element  $b_{12} \in PB_3$ . Therefore,

near  $(a_1, a_2)$ , the series (3.23) is convergent to  $f_{\mathcal{Y},2}^{b_{12}(b_{13}b_{12})^{p_1}b_{23}^{p_2}}(z_1, z_2; u, w_1, w_2, w'_3)$ .



□

**Lemma 3.0.34.** For any  $a \in \pi_1(F_2(\mathbb{C}^\times))$ ,  $u \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w'_3 \in W'_3$ , we have

$$f_{\mathcal{Y},2}^a(z_1, z_2; u, w_1, w_2, w'_3) \quad (3.34)$$

$$= f_{\mathcal{Y},2}^{b_{12}a}(z_1, z_2; g_1u, w_1, w_2, w'_3) \quad (3.35)$$

$$= f_{\mathcal{Y},2}^{b_{12}^{-1}b_{13}b_{12}a}(z_1, z_2; g_2u, w_1, w_2, w'_3) \quad (3.36)$$

$$= f_{\mathcal{Y},2}^{b_{13}b_{12}a}(z_1, z_2; g_3u, w_1, w_2, w'_3) \quad (3.37)$$

*Proof.* We only prove that (3.34) is equal to (3.35), since the others are similar.

We only need to prove the case when  $a = e$ , because if  $f_{\mathcal{Y},2}^{b_{12}}(z_1, z_2; g_1u, w_1, w_2, w'_3) = f_{\mathcal{Y},2}^e(z_1, z_2; u, w_1, w_2, w'_3)$ , doing analytic continuation on the both side along the path corresponding to  $a \in BP_3$ , one gets that (3.34) is equal to (3.35). The following steps are similar to the proof of lemma 4.6 in [H9]. On the region given by  $|z_2| > |z_1 - z_2| > 0$ , and  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ , by the duality of  $\mathcal{Y}$ , the series

$$\begin{aligned} & \langle w'_3, \mathcal{Y}^0 \left( (Y_{W_1}^{g_1})^1(g_1u, z_1 - z_2)w_1, z_2 \right) w_2 \rangle, \\ & \langle w'_3, \mathcal{Y}^0 \left( (Y_{W_1}^{g_1})^0(u, z_1 - z_2)w_1, z_2 \right) w_2 \rangle \end{aligned} \quad (3.38)$$

are absolutely convergent to  $f_{\mathcal{Y},2}^{b_{12}}(z_1, z_2; g_1u, w_1, w_2, w'_3)$  and  $f_{\mathcal{Y},2}^e(z_1, z_2; u, w_1, w_2, w'_3)$

respectively. By the equivariance property of  $Y_{W_1}^{g_1}$ , we know the above two series are equal. Therefore, we know that  $f_{\mathcal{Y},2}^{b_{12}}(z_1, z_2; g_1u, w_1, w_2, w'_3) = f_{\mathcal{Y},2}^e(z_1, z_2; u, w_1, w_2, w'_3)$ .

□

Let  $F\langle x_1, x_2, x_3 \rangle$  be the free group with generators  $x_1, x_2, x_3$ . Define the group homomorphism  $\varphi : F\langle x_1, x_2, x_3 \rangle \rightarrow \pi_1(F_2(\mathbb{C}^\times))$  by  $\varphi(x_1) = b_{12}$ ,  $\varphi(x_2) = b_{12}^{-1}b_{13}b_{12}$ ,  $\varphi(x_3) = b_{13}b_{12}$ . Define the group homomorphism  $\psi : F\langle x_1, x_2, x_3 \rangle \rightarrow \langle g_1, g_2, g_3 \rangle$  by  $\psi(x_i) = g_i$ ,  $i = 1, 2, 3$ . Then, by lemma 3.0.34, we have

**Corollary 3.0.35.** *For any  $a \in \pi_1(F_2(\mathbb{C}^\times))$  and  $x \in F\langle x_1, x_2, x_3 \rangle$ , we have*

$$f_{\mathcal{Y},2}^a(z_1, z_2; \psi(x)u, w_1, w_2, w'_3) = f_{\mathcal{Y},2}^{\varphi(x)^{-1}a}(z_1, z_2; u, w_1, w_2, w'_3) \quad (3.39)$$

**Theorem 3.0.36.** *Let  $W_i$  be  $g_i$ -twisted modules for  $i = 1, 2, 3$ . Assume that any one of the following conditions holds:*

- (a) *The map  $u \mapsto Y_{W_3}^{g_3}(u, x)$  is injective. There exists a twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1W_2}$  such that the coefficients of the series  $\mathcal{Y}(w_1, x)w_2$  span  $W_3$ , as  $w_1, w_2$  range through  $W_1, W_2$ , respectively.*
- (b) *The map  $u \mapsto Y_{W_1}^{g_1}(u, x)$  is injective. There exists a twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1W_2}$  such that the map  $w_1 \mapsto \mathcal{Y}(w_1, x)$  is injective.*
- (c) *The map  $u \mapsto Y_{W_2}^{g_2}(u, x)$  is injective. There exists a twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1W_2}$  such that the map  $w_2 \mapsto \mathcal{Y}(\cdot, x)w_2$  is injective.*

Then, we have  $g_3 = g_1g_2$ .

Let (a') be the conditions obtained from the condition (a) by dropping the requirement that  $u \mapsto Y_{W_3}^{g_3}(u, x)$  is injective. Then, if the condition (a') holds,  $W_3$  is a  $g_1g_2$ -twisted module.

*Proof.* We first prove (a). By lemma 2.7, we know that

$$\begin{aligned}
& f_{\mathcal{Y},2}^{b_{13}b_{12}}(z_1, z_2; g_3u, w_1, w_2, w'_3) \\
&= f_{\mathcal{Y},2}^e(z_1, z_2; u, w_1, w_2, w'_3) \\
&= f_{\mathcal{Y},2}^{b_{12}^{-1}b_{13}b_{12}}(z_1, z_2; g_2u, w_1, w_2, w'_3) \\
&= f_{\mathcal{Y},2}^{b_{12} \cdot (b_{12}^{-1}b_{13}b_{12})}(z_1, z_2; g_1g_2u, w_1, w_2, w'_3) \\
&= f_{\mathcal{Y},2}^{b_{13}b_{12}}(z_1, z_2; g_1g_2u, w_1, w_2, w'_3)
\end{aligned} \tag{3.40}$$

When  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ , the left-hand side and the right-hand side of (3.40) are absolutely convergent to  $\langle w'_3, (Y_{W_3}^{g_3})^1(g_3u, z_1)\mathcal{Y}^0(w_1, z_2)w_2 \rangle$  and  $\langle w'_3, (Y_{W_3}^{g_3})^1(g_1g_2u, z_1)\mathcal{Y}^0(w_1, z_2)w_2 \rangle$ , respectively.

$$\langle w'_3, (Y_{W_3}^{g_3})^1(g_1g_2u - g_3u, z_1)\mathcal{Y}^0(w_1, z_2)w_2 \rangle = 0,$$

for any  $w_1 \in W_1, w_2 \in W_2, w_3 \in W'_3$ . Therefore, condition (a') implies that  $Y_{W_3}^{g_3}(g_1g_2u - g_3u, x) = 0$  for any  $u \in V$ . Since  $W_3$  is  $g_3$ -twisted, we know that it is also  $g_1g_2$ -twisted. If we further know that  $Y_{W_3}^{g_3}(\cdot, x)$  is injective, then,  $g_1g_2 = g_3$ . For (b) and (c), notice that we can obtain the followings similarly on some other regions.

$$\langle w'_3, \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(g_1g_2u - g_3u, z_1 - z_2)w_1, z_2)w_2 \rangle = 0,$$

$$\langle w'_3, \mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_2}^{g_2})^{p_1}(g_1g_2u - g_3u, z_1)w_2 \rangle = 0.$$

□

**Lemma 3.0.37.** For  $n \in \mathbb{Z}_+$ ,  $u_1, \dots, u_{n-1} \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $w'_3 \in W'_3$ , the series

$$\langle w'_3, \mathcal{Y}(w_1, z_n)Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1})w_2 \rangle$$

is absolutely convergent on the region  $|z_n| > |z_1| > \cdots > |z_{n-1}| > 0$ . It is equal to  $f_{\mathcal{Y},n}^e(z_1, \dots, z_n)$  on the region given by  $|z_n| > |z_1| > \cdots > |z_{n-1}| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_i - z_n) - \arg z_n < -\frac{\pi}{2}$ ,  $i = 1, \dots, n-1$ .

*Proof.* We use induction on  $n$ . For  $n \leq 2$ , the statement holds due to the duality of  $\mathcal{Y}$ . Now we shall prove for  $n > 2$  case. Write  $P = \prod_{1 \leq i < j < n} (z_i - z_j)^M$ ,  $Q = \prod_{2 \leq i < j < n} (z_i - z_j)^M$ , and  $R = \prod_{2 \leq j < n} (z_1 - z_j)^M$ , where  $M \in \mathbb{Z}_+$  is sufficiently large. We know that on the region given by  $|z_1| > \dots > |z_n| > 0$ ,  $|\arg(z_i - z_n) - \arg z_i| < \frac{\pi}{2}$ ,  $i = 1, \dots, n-1$ ,

$$\begin{aligned}
& P \cdot f_{\mathcal{Y},n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3) \\
&= P \cdot \langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) \mathcal{Y}(w_1, z_n) w_2 \rangle \\
&= P \cdot \sum_{m_1 \in D} \sum_{k_1 \in \mathbb{N}} \left\langle (Y_{W_3}^{g_3})_{-m_1-1, k_1}^o(u_1) w'_3, Y_{W_3}^{g_3}(u_2, z_2) \cdots \right. \\
&\quad \left. Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) \mathcal{Y}(w_1, z_n) w_2 \right\rangle z_1^{m_1} \log(z_1)^{k_1} \\
&= P \cdot \sum_{m_1 \in D} \sum_{k_1 \in \mathbb{N}} f_{\mathcal{Y},n-1}^e(z_2, \dots, z_n; u_2, \dots, u_{n-1}, w_1, w_2, (Y_{W_3}^{g_3})_{-m_1-1, k_1}^o(u_1) w'_3) z_1^{m_1} \log(z_1)^{k_1} \\
&= R \cdot \sum_{m_1 \in D} \sum_{k_1 \in \mathbb{N}} \left( Q \cdot f_{\mathcal{Y},n-1}^e(z_2, \dots, z_n; u_2, \dots, u_{n-1}, w_1, w_2, \right. \\
&\quad \left. (Y_{W_3}^{g_3})_{-m_1-1, k_1}^o(u_1) w'_3) \right) z_1^{m_1} \log(z_1)^{k_1} \tag{3.41}
\end{aligned}$$

is absolutely convergent, where  $D \subset \mathbb{C}$  is a finite union of sets of the form  $a - \mathbb{N}$ ,  $a \in \mathbb{C}$ . We know that

$$\begin{aligned}
& P \cdot f_{\mathcal{Y},n}(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3), \tag{3.42} \\
& Q \cdot f_{\mathcal{Y},n-1}(z_2, \dots, z_n; u_2, \dots, u_{n-1}, w_1, w_2, (Y_{W_3}^{g_3})_{-m_1-1, k_1}^o(u_1) w'_3)
\end{aligned}$$

can be analytic extended to

$$\{(z_1, \dots, z_n) \in (\mathbb{C} - \{0\})^n \mid z_n \neq z_i, i = 1, \dots, n-1\},$$

$$\{(z_2, \dots, z_n) \in (\mathbb{C} - \{0\})^{n-1} | z_n \neq z_i, i = 2, \dots, n-1\}, \quad (3.43)$$

respectively. For any  $(\tilde{z}_2, \dots, \tilde{z}_n) \in (3.43)$ , by part (c) in Condition 3.0.13, we know that  $z_1 = \infty$  is a regular singularity of  $P \cdot f_{y,n}(z_1, \tilde{z}_2, \dots, \tilde{z}_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3)$ . Therefore, We know that on the region given by  $|z_1| > |z_n|$ , and  $(z_2, \dots, z_n) \in (3.43)$ , the right-hand side of (3.41) is well-defined and absolutely convergent to (3.42). Moreover, the right-hand side is equal to the left-hand side of (3.41) on the region given by  $|z_1| > |z_n|$ ,  $(z_2, \dots, z_n) \in (3.43)$ , and  $|\arg(z_1 - z_n) - \arg z_1| < \frac{\pi}{2}$ .

Therefore, on the region  $|z_1| > |z_n| > |z_i| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_i - z_n) - \arg z_n < -\frac{\pi}{2}$ ,  $i = 2, \dots, n-1$ , using the induction hypothesis, we know that the right-hand side, and therefore the left-hand side, of (3.41) is equal to

$$\begin{aligned} & R \cdot \sum_{m_1 \in D} \sum_{k_1 \in \mathbb{N}} Q \left\langle (Y_{W_3}^{g_3})_{-m_1-1, k_1}^o(u_1) w'_3, \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_2, z_2) \cdots \right. \\ & \quad \left. Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \right\rangle z_1^{m_1} \log(z_1)^{k_1} \\ & = P \cdot \left\langle w'_3, Y_{W_3}^{g_3}(u_1, x_1) \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_2, z_2) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \right\rangle \Big|_{x_1=z_1}. \end{aligned} \quad (3.44)$$

Since for any  $\tilde{z}_n \in \mathbb{C} - \{0\}$ ,  $(z_1, \dots, z_{n-1}) = (\infty, 0, \dots, 0)$  is a regular singularity of the function obtained from (3.42) by substituting  $z_n = \tilde{z}_n$ , we know that

$$P \cdot \left\langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_2, z_2) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \right\rangle \quad (3.45)$$

is absolutely convergent to  $P \cdot f_{y,n}(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2, w'_3)$  on the region  $|z_1| > |z_n| > |z_i| > 0$ . Moreover, on the region given by  $|z_1| > |z_n| > |z_i| > 0$ ,  $|\arg(z_1 - z_n) - \arg z_1| < \frac{\pi}{2}$ ,  $-\frac{3\pi}{2} < \arg(z_i - z_n) - \arg z_n < -\frac{\pi}{2}$ ,  $i = 2, \dots, n-1$

$$\begin{aligned} & P \cdot \left\langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_2, z_2) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \right\rangle \\ & = P \cdot f_{y,n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3) \end{aligned}$$

Similar as above, on the region given by  $|z_1| > |z_n| > |z_i| > 0$ ,  $|\arg(z_1 - z_n) - \arg z_1| < \frac{\pi}{2}$ ,  $i = 2 \dots, n - 1$ , we have

$$\begin{aligned}
& P \cdot \langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_2, z_2) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \rangle \\
&= P \cdot \sum_{m_2, \dots, m_{n-1} \in \mathbb{C}} \sum_{k_2, \dots, k_{n-1} \in \mathbb{N}} \left\langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \mathcal{Y}(w_1, z_n) (Y_{W_3}^{g_3})_{-m_2-1, k_2}(u_2) \cdots \right. \\
&\quad \left. (Y_{W_3}^{g_3})_{-m_{n-1}-1, k_{n-1}}(u_{n-1}) w_2 \right\rangle z_2^{m_2} \log(z_2)^{k_2} \cdots z_{n-1}^{m_{n-1}} \log(z_{n-1})^{k_{n-1}} \\
&= P \cdot \sum_{m_2, \dots, m_{n-1} \in \mathbb{C}} \sum_{k_2, \dots, k_{n-1} \in \mathbb{N}} f_{\mathcal{Y}, 2}^e \left( z_1, z_n; u_1, w_1, (Y_{W_3}^{g_3})_{-m_2-1, k_2}(u_2) \cdots \right. \\
&\quad \left. (Y_{W_3}^{g_3})_{-m_{n-1}-1, k_{n-1}}(u_{n-1}) w_2, w'_3 \right) z_2^{m_2} \log(z_2)^{k_2} \cdots z_{n-1}^{m_{n-1}} \log(z_{n-1})^{k_{n-1}}
\end{aligned} \tag{3.46}$$

Since for any  $(\tilde{z}_1, \tilde{z}_n) \in F_2(\mathbb{C} - \{0\})$ ,  $(z_2, \dots, z_{n-1}) = (0, \dots, 0)$  is a regular singularity of the function obtained from (3.42) by substituting  $z_1 = \tilde{z}_1$  and  $z_n = \tilde{z}_n$ , the right-hand side of (3.46) is well-defined and absolutely convergent to (3.42) on the region given by  $(z_1, z_n) \in F_2(\mathbb{C} - 0)$ ,  $|z_n| > |z_i| > 0$ ,  $i = 2, \dots, n - 1$ . Moreover, on the region given by  $(z_1, z_n) \in F_2(\mathbb{C} - 0)$ ,  $|z_n| > |z_i| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_i - z_n) - \arg z_n < -\frac{\pi}{2}$ ,  $i = 2 \dots, n - 1$ , the right-hand side of (3.46) is equal to  $P \cdot f_{\mathcal{Y}, n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3)$ . Therefore, on the region given by  $|z_n| > |z_i| > 0$ ,  $i = 1, \dots, n - 1$ ,  $-\frac{3\pi}{2} < \arg(z_1 - z_n) - \arg z_n < -\frac{\pi}{2}$ , we know that the right-hand side of (3.46) is equal to

$$\begin{aligned}
& P \cdot \sum_{m_1, \dots, m_{n-2} \in \mathbb{C}} \sum_{k_1, \dots, k_{n-2} \in \mathbb{N}} \left\langle w'_3, \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_1, z_1) (Y_{W_3}^{g_3})_{-m_2-1, k_2}(u_2) \cdots \right. \\
&\quad \left. (Y_{W_3}^{g_3})_{-m_{n-1}-1, k_{n-1}}(u_{n-1}) w_2 \right\rangle z_2^{m_2} \log(z_2)^{k_2} \cdots z_{n-1}^{m_{n-1}} \log(z_{n-1})^{k_{n-1}}.
\end{aligned} \tag{3.47}$$

Since  $(z_1, \dots, z_n) = (0, \dots, 0, \infty)$  is a regular singularity of (3.42), we know that

$$P \cdot \langle w'_3, \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \rangle$$

is absolutely convergent to (3.42) on the region given by  $|z_n| > |z_i| > 0$   $i = 1, \dots, n-1$ . Moreover, on the region given by  $|z_n| > |z_i| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_i - z_n) - \arg z_n < -\frac{\pi}{2}$ ,  $i = 1, \dots, n-1$ , by (3.46) and (3.47), we have

$$\begin{aligned}
& P \cdot \langle w'_3, \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \rangle \\
&= P \cdot \sum_{m_1, \dots, m_{n-2} \in \mathbb{C}} \sum_{k_1, \dots, k_{n-2} \in \mathbb{N}} \left\langle w'_3, \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_1, z_1) (Y_{W_3}^{g_3})_{-m_2-1, k_2}(u_2) \cdots \right. \\
&\quad \left. (Y_{W_3}^{g_3})_{-m_{n-1}-1, k_{n-1}}(u_{n-1}) w_2 \right\rangle z_2^{m_2} \log(z_2)^{k_2} \cdots z_{n-1}^{m_{n-1}} \log(z_{n-1})^{k_{n-1}}. \\
&= P \cdot \sum_{m_1, \dots, m_{n-2} \in \mathbb{C}} \sum_{k_1, \dots, k_{n-2} \in \mathbb{N}} f_{\mathcal{Y}, 2}^e(z_1, z_n; u_1, w_1, (Y_{W_3}^{g_3})_{-m_2-1, k_2}(u_2) \cdots \\
&\quad (Y_{W_3}^{g_3})_{-m_{n-1}-1, k_{n-1}}(u_{n-1}) w_2, w'_3) z_2^{m_2} \log(z_2)^{k_2} \cdots z_{n-1}^{m_{n-1}} \log(z_{n-1})^{k_{n-1}} \\
&= P \cdot f_{\mathcal{Y}, n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3) \tag{3.48}
\end{aligned}$$

Multiplying

$$\prod_{1 \leq i < j < n} (\iota_{ij}(z_i - z_j)^{-1})^M, \text{ where } \iota_{ij}(z_i - z_j)^{-1} = \sum_{k \in \mathbb{N}} z_i^{-k-1} z_j^k,$$

which is absolutely convergent on the region given by  $|z_1| > \dots > |z_{n-1}| > 0$ , on the both sides of (3.48), we know that

$$\langle w'_3, \mathcal{Y}(w_1, z_n) Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{n-1}, z_{n-1}) w_2 \rangle$$

is absolutely convergent on the region given by  $|z_n| > |z_1| > \dots > |z_{n-1}| > 0$ . Moreover, it is equal to  $f_{\mathcal{Y}, n}^e(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3)$  on the region given by  $|z_n| > |z_1| > \dots > |z_{n-1}| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_i - z_n) - \arg z_n < -\frac{\pi}{2}$ ,  $i = 1, \dots, n-1$ .  $\square$

**Remark 3.0.38.** *Similar as Remark 2.0.3, the following result relies on the fact that*

any set of the form

$$\left(\cup_{i=1}^N (r_i + \mathbb{Z})\right) \times \{0, \dots, N'\},$$

where  $N \in \mathbb{Z}_+$ ,  $r_i \in \mathbb{C}$ ,  $i = 1, \dots, N$ , and  $N' \in \mathbb{N}$ , is a unique expansion set.

On the region  $|z_i| > |z_n| > 0$ ,  $|\arg(z_i - z_n) - \arg z_i| < \frac{\pi}{2}$ ,  $i = 1, \dots, n-1$ , for sufficient large  $M \in \mathbb{Z}_+$ , we have an expansion

$$\begin{aligned} & \prod_{1 \leq i < j < n} (z_i - z_j)^M f_{\mathcal{Y}, n}^{fe}(z_1, \dots, z_n; u_1, \dots, u_{n-1}, w_1, w_2, w'_3) \\ &= \sum_{m_1, \dots, m_n \in \mathbb{C}} \sum_{k_1, \dots, k_n \in \mathbb{N}} a_{m_1, k_1; \dots; m_n, k_n} z_1^{m_1} \log(z_1)^{k_1} \cdots z_n^{m_n} \log(z_n)^{k_n}. \end{aligned}$$

Therefore, by Remark 3.0.24, we know that

$$\begin{aligned} & \prod_{1 \leq i < j < n} (x_i - x_j)^M \langle w'_3, Y_{W'_3}^{g_3}(u_1, x_1) \cdots Y_{W'_3}^{g_3}(u_{n-1}, x_{n-1}) \mathcal{Y}(w_1, x_n) w_2 \rangle \\ &= \sum_{m_1, \dots, m_n \in \mathbb{C}} \sum_{k_1, \dots, k_n \in \mathbb{N}} a_{m_1, k_1; \dots; m_n, k_n} x_1^{m_1} \log(x_1)^{k_1} \cdots x_n^{m_n} \log(x_n)^{k_n}, \end{aligned}$$

where  $x_1, \dots, x_n$  are formal variables.

**Remark 3.0.39.** Using duality and Cauchy formula, we can prove that for any  $v \in V^{\langle g_1, g_2, g_3 \rangle}$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w_3 \in W'_3$ , the following Jacobi identity holds (see Proposition 7.0.10).

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle w'_3, Y_{W'_3}^{g_3}(v, x_1) \mathcal{Y}(w_1, x_2) w_2 \rangle \\ & \quad - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle w'_3, \mathcal{Y}(w_1, x_2) Y_{W'_3}^{g_2}(v, x_1) w_2 \rangle \\ &= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \langle w'_3, \mathcal{Y}(Y_{W_1}^{g_1}(v, x_0) w_1, x_2) w_2 \rangle \end{aligned} \quad (3.49)$$

In the sense that the coefficients of each formal monomial in both sides of (3.49) is

absolutely convergent. Taking  $\text{Res}_{x_0}$ , we get the commutator formula,

$$\begin{aligned} \langle w'_3, Y_{W_3}^{g_3}(v, x_1)\mathcal{Y}(w_1, x_2)w_2 \rangle - \langle w'_3, \mathcal{Y}(w_1, x_2)Y_{W_2}^{g_2}(v, x_1)w_2 \rangle = \\ \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \langle w'_3, \mathcal{Y}(Y_{W_1}^{g_1}(v, x_0)w_1, x_2)w_2 \rangle. \end{aligned} \quad (3.50)$$

Let  $v$  in (3.50) be  $\omega \in V^{\text{Aut}(V)} \subset V^{(g_1, g_2, g_3)}$ . Then, we have the Virasoro-bracket relation,

$$\begin{aligned} \langle w'_3, L_{W_3}(m)\mathcal{Y}(w_1, x)w_2 \rangle - \langle w'_3, \mathcal{Y}(w_1, x)L_{W_2}(m)w_2 \rangle \\ = \sum_{k \in \mathbb{N}} \binom{m+1}{k} x^{m-k+1} \langle w'_3, \mathcal{Y}(L_{W_1}(k-1)w_1, x)w_2 \rangle, \text{ for any } m \in \mathbb{Z}, w_1 \in W_1. \end{aligned} \quad (3.51)$$

Using  $L(0)$ -commutator formula (the  $m = 0$  piece in (3.51), where we can drop the pairing  $\langle w'_3, \cdot w_2 \rangle$  since there are finitely many terms on the right-hand side),

$$L_{W_3}(0)\mathcal{Y}(w_1, x) - \mathcal{Y}(w_1, x)L_{W_2}(0) = x\mathcal{Y}(L_{W_1}(-1)w_1, x) + \mathcal{Y}(L_{W_1}(0)w_1, x),$$

the  $L(0)$ -grading structure on  $W_i$ ,  $i = 1, 2, 3$ , and the  $L(-1)$ -derivative formula of  $\mathcal{Y}$ , we can prove the compatibility between the  $L(0)$ -grading and component of  $\mathcal{Y}$ , i.e. for any  $n \in \mathbb{C}$ ,  $k \in \mathbb{N}$ , and homogeneous  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,

$$\text{wt} \left( (\mathcal{Y})_{n,k}(w_1)(w_2) \right) = \text{wt } w_1 + \text{wt } w_2 - n - 1. \quad (3.52)$$

(See Proposition 3.20 in [HLZ2]). Also, as a result of (3.51), the  $L(0)$ -grading structure on  $W_i$ ,  $i = 1, 2, 3$ , and the  $L(-1)$ -derivative formula of  $\mathcal{Y}$ , we can prove that

$$\begin{aligned} e^{yL(-1)}\mathcal{Y}(w_1, x)e^{-yL(-1)} &= \mathcal{Y}(e^{yL(-1)}w_1, x) = \mathcal{Y}(w_1, x + y), \\ y^{L(0)}\mathcal{Y}(w_1, x)y^{-L(0)} &= \mathcal{Y}(y^{L(0)}w_1, xy), \end{aligned}$$

$$\langle w'_3, e^{yL(1)}\mathcal{Y}(w_1, x)e^{-yL(1)}w_2 \rangle = \langle w'_3, \mathcal{Y}(e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w_1, x(1-yx)^{-1})w_2 \rangle.$$

(See Proposition 3.36 in [HLZ2]) Notice that the third identity holds only in the sense that the coefficient of each formal monomial on both sides are absolutely convergent. This is because  $L(1)$  is not necessarily locally nilpotent. (We would require this from the next section.)

## Chapter 4

# Skew-symmetry Isomorphisms $\Omega_{\pm}$

To make the skew-symmetry isomorphism  $\Omega_{\pm}$  and the contragredient isomorphism  $A_{\pm}$  well-defined, we make the following assumption.

**Assumption 4.0.1.** *Throughout the remainder of this work, for any twisted module  $(W, Y_W)$  that is considered, we assume that the following conditions hold:*

1. *Let  $L_W^g(0)_N = L(0)_N$  be the nilpotent part of  $L_W^g(0)$ . Then, there exists  $K_W \in \mathbb{N}$ , such that  $L(0)_N^{K_W} = 0$ . In other words, the locally nilpotent operator  $L(0)_N$  is actually nilpotent on  $W$ .*
2.  *$L(1)_W$  and  $L(1)_{W'}$  are locally nilpotent. (See Remark 2.0.5)*

**Remark 4.0.2.** *Part 2. of Assumption 4.0.1 looks technical. This is because we want to put the theory in a setting as general as possible. Note that if all the modules that we consider are lower-bounded, then part 2 in Assumption 4.0.1 trivially holds.*

**Remark 4.0.3.** *By part 1. of Assumption 4.0.1, and the  $L(0)$ -conjugation formula, for any twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1 W_2}$ , we know that*

$$\max\{k \in \mathbb{N} \mid \mathcal{Y}_{n,k} \neq 0 \text{ for some } n \in \mathbb{C}\} \leq K_{W_1} + K_{W_2} + K_{W_3} < \infty,$$

*which is needed to make  $A_{\pm}$  well-defined (See section 4).*

Let  $g_1, g_2$  be automorphisms of  $V$ ,  $W_1$ ,  $W_2$  and  $W_3$   $g_1$ -,  $g_2$ - and  $g_1g_2$ -twisted  $V$ -modules and  $\mathcal{Y}$  a twisted intertwining operator of type  $\left(\begin{smallmatrix} W_3 \\ W_1W_2 \end{smallmatrix}\right)$ . We define linear maps

$$\begin{aligned} \Omega_{\pm}(\mathcal{Y}) : W_2 \otimes W_1 &\rightarrow W_3\{x\}[\log x] \\ w_2 \otimes w_1 &\mapsto \Omega_{\pm}(\mathcal{Y})(w_2, x)w_1 \end{aligned} \quad (4.1)$$

by

$$\langle w'_3, \Omega_{\pm}(\mathcal{Y})(w_2, x)w_1 \rangle = \left\langle e^{xL(1)_{W'_3}} w'_3, \mathcal{Y}(w_1, y)w_2 \right\rangle \Big|_{y^n = e^{\pm n\pi i} x^n, n \in \mathbb{C}, \log y = \log x \pm \pi i} \quad (4.2)$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . Notice that (4.2) is well-defined because by part 2 in Assumption 4.0.1,  $e^{xL(1)_{W'_3}} w'_3$  is a polynomial for any  $w'_3$ .

We have the component form, for any  $n \in \mathbb{C}$ ,  $k \in \mathbb{N}$ ,

$$\Omega_{\pm}(\mathcal{Y})_{n,k}(w_2)w_1 = \sum_{l \in \mathbb{N}} \sum_{j \geq k} \frac{1}{l!} \binom{j}{k} e^{\mp(l+n+1)\pi i} (\pm\pi i)^{j-k} L(-1)^l \mathcal{Y}_{n+l,j}(w_1)w_2. \quad (4.3)$$

The powers of  $\log(x)$  in  $\Omega_{\pm}(\mathcal{Y})(w_2, x)w_1$  have an upper bound, because of the definition (4.2) and the fact that this holds for  $\mathcal{Y}(w_1, x)w_2$ .

From the definition (4.2), for  $p \in \mathbb{Z}$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $z \in \mathbb{C}^{\times}$ ,

$$\begin{aligned} \Omega_{\pm}(\mathcal{Y})^p(w_2, z)w_1 &= \Omega_{\pm}(\mathcal{Y})(w_2, x)w_1 \Big|_{x^n = e^{nlp(z)}, \log x = lp(z)} \\ &= \left( e^{xL(-1)} \mathcal{Y}(w_1, y)w_2 \Big|_{y^n = e^{\pm n\pi i} x^n, \log y = \log x \pm \pi i} \right) \Big|_{x^n = e^{nlp(z)}, \log x = lp(z)} \\ &= e^{zL(-1)} \mathcal{Y}(w_1, y)w_2 \Big|_{y^n = e^{n(lp(z) \pm \pi i)}, \log y = lp(z) \pm \pi i}. \end{aligned}$$

When  $\arg z < \pi$  and  $\arg z \geq \pi$ ,  $\arg(-z) = \arg z + \pi$  and  $\arg(-z) = \arg z - \pi$ ,

respectively. Hence

$$e^{zL(-1)}\mathcal{Y}(w_1, y)w_2 \Big|_{y^n=e^{n(l_p(z)+\pi i)}, \log y=l_p(z)+\pi i} = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2$$

when  $\arg z < \pi$  and

$$e^{zL(-1)}\mathcal{Y}(w_1, y)w_2 \Big|_{y^n=e^{n(l_p(z)-\pi i)}, \log y=l_p(z)-\pi i} = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2$$

when  $\arg z \geq \pi$ . In particular, for  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $z \in \mathbb{C}^\times$  satisfying  $\arg z < \pi$  and  $\arg z \geq \pi$ , we have

$$\Omega_+(\mathcal{Y})^p(w_2, z)w_1 = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2 \quad (4.4)$$

and

$$\Omega_-(\mathcal{Y})^p(w_2, z)w_1 = e^{zL(-1)}\mathcal{Y}^p(w_1, -z)w_2, \quad (4.5)$$

respectively.

**Theorem 4.0.4.** *The linear maps  $\Omega_+(\mathcal{Y})$  and  $\Omega_-(\mathcal{Y})$  are twisted intertwining operators of types  $({}_{W_2\phi_{g_2^{-1}}(W_1)}^{W_3})$  and  $({}_{\phi_{g_1}(W_2)W_1}^{W_3})$ , respectively. If  $\mathcal{Y}$  is fully lower truncated (or lower truncated), then  $\Omega_\pm(\mathcal{Y})$  also fully lower truncated (or lower truncated).*

*Proof.* The second sentence in this theorem is clear by the definition of  $\Omega_\pm(\mathcal{Y})$ . Now we prove the first sentence.

We shall start with the duality property.

Let  $u \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w_3 \in W'_3$ . We consider

$$\begin{aligned} & \langle w'_3, (Y_{W_3}^{g_3})^0(u, z_1)\Omega_+(\mathcal{Y})^0(w_2, z_2)w_1 \rangle \\ &= \langle w'_3, Y_{W_3}^{g_3}(u, x)\Omega_+(\mathcal{Y})(w_2, y)w_1 \rangle \Big|_{\substack{x^n=e^{nl_0(z_1)}, \log(x)=l_0(z_1) \\ y^n=e^{nl_0(z_2)}, \log(y)=l_0(z_2)}} \end{aligned}$$

$$\begin{aligned}
&= \left( \langle w'_3, Y_{W_3}^{g_3}(u, x) e^{yL(-1)} \mathcal{Y}(w_1, \tilde{y}) w_2 \rangle \Big|_{\substack{\tilde{y}^n = e^{n\pi i} y^n, \log(\tilde{y}) = \log(y) + \pi i}} \right) \Big|_{\substack{x^n = e^{n l_0(z_1)}, \log(x) = l_0(z_1) \\ y^n = e^{n l_0(z_2)}, \log(y) = l_0(z_2)}} \\
&= \langle w'_3, Y_{W_3}^{g_3}(u, x) e^{-\tilde{y}L(-1)} \mathcal{Y}(w_1, \tilde{y}) w_2 \rangle \Big|_{\substack{x^n = e^{n l_0(z_1)}, \log(x) = l_0(z_1) \\ \tilde{y}^n = e^{n\pi i} e^{n l_0(z_2)}, \log(\tilde{y}) = l_0(z_2) + \pi i}} \\
&= \langle e^{z_2 L(1)} w'_3, Y_{W_3}^{g_3}(u, x + \tilde{y}) \mathcal{Y}(w_1, \tilde{y}) w_2 \rangle \Big|_{\substack{x^n = e^{n l_0(z_1)}, \log(x) = l_0(z_1) \\ \tilde{y}^n = e^{n\pi i} e^{n l_0(z_2)}, \log(\tilde{y}) = l_0(z_2) + \pi i}} . \tag{4.6}
\end{aligned}$$

Let  $q_1 = q_1(z_1, z_2) = -1, 0, 1$  if  $\arg(z_1 - z_2) - \arg z_1 \in (\frac{3\pi}{2}, 2\pi), (-\frac{\pi}{2}, \frac{\pi}{2}), (-2\pi, -\frac{3\pi}{2})$ , respectively. Let  $q_2 = q_2(z_2) = 0, 1$  if  $\arg z_2 < \pi$ , or  $\arg z_2 \geq \pi$ , respectively. Notice that for any  $s \in \mathbb{C}$ , on the region  $|z_1| > |z_2| > 0$ , we have

$$\begin{aligned}
e^{s l_{q_1}(z_1 - z_2)} &= \sum_{k \geq 0} \binom{s}{k} (-1)^k e^{(s-k)l_0(z_1)} z_2^k, \\
l_{q_1}(z_1 - z_2) &= l_0(z_1) + \sum_{k > 0} \frac{-1}{k} z_1^{-k} z_2^k, \\
e^{s l_{q_2}(-z_2)} &= e^{s\pi i} e^{s l_0(z_2)}, \quad l_{q_2}(-z_2) = l_0(z_2) + \pi i. \tag{4.7}
\end{aligned}$$

We know that the series (in variables  $z_1 - z_2$  and  $z_2$ )

$$\langle e^{z_2 L(1)} w'_3, (Y_{W_3}^{g_3})^{q_1}(u, z_1 - z_2) \mathcal{Y}^{q_2}(w_1, -z_2) w_2 \rangle \tag{4.8}$$

is absolutely convergent to

$$\left\{ \begin{array}{l} f_{\mathcal{Y}, 2}^{a^{q_1}, q_2 b_{12}}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3), \\ \quad \text{if } |z_1 - z_2| > |z_2| > 0, \quad 3\pi/2 < \arg(z_1 - z_2) - \arg z_1 < 2\pi \\ f_{\mathcal{Y}, 2}^{a^{q_1}, q_2}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3), \\ \quad \text{if } |z_1 - z_2| > |z_2| > 0, \quad |\arg(z_1 - z_2) - \arg z_1| < \pi/2 \\ f_{\mathcal{Y}, 2}^{a^{q_1}, q_2 b_{12}^{-1}}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3), \\ \quad \text{if } |z_1 - z_2| > |z_2| > 0, \quad -2\pi < \arg(z_1 - z_2) - \arg z_1 < -3\pi/2. \end{array} \right. \tag{4.9}$$

Let  $h_+(z_1, z_2; u, w_1, w_2, w'_3)$  be the multivalued holomorphic function on  $F_2(\mathbb{C}^\times)$  with a preferred single-valued branch  $h_+^e$ , such that  $h_+^e(z_1, z_2; u, w_1, w_2, w'_3)$  is equal to (4.8) on the region given by  $|z_1| > |z_2| > 0$ ,  $|z_1 - z_2| > |z_2|$ , and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ . By (4.9), we know that

$$h_+^e(z_1, z_2; u, w_1, w_2, w'_3) = f_{\mathcal{Y}, 2}^{a_{q_1, q_2}}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3) \quad (4.10)$$

when  $|z_1| > |z_2| > 0$ ,  $|z_1 - z_2| > |z_2| > 0$ , and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ .

By appendix A, we know that the right-hand side of (4.6) is absolutely convergent to (4.8) on the region given by  $|z_1|/2 > |z_2| > 0$ . However, the only singularities of (4.9) are  $z_1, z_2, z_1 - z_2 = 0$ , which means that (4.8) is a multi-valued holomorphic function in the region given by  $|z_1| > |z_2| > 0$ . So, (4.6) can be analytically extended from the region given by  $|z_1|/2 > |z_2| > 0$  to the region given by  $|z_1| > |z_2| > 0$ . Also, in the region given by  $|z_1|/2 > |z_2| > 0$ , (4.6) has the form of (3.30). The fact that (4.6) can be analytically extended to the region given by  $|z_1| > |z_2| > 0$  implies that those power series  $f_{\mu, i, j}$  in (3.30) have the radius of convergence not less than 1. So, the left-hand side of (4.6) is absolutely convergent in the region given by  $|z_1| > |z_2| > 0$ . By the definition of  $h_+$ , (4.6) is absolutely convergent to  $h_+^e(z_1, z_2, u, w_1, w_2, w_3)$  on the region given by  $|z_1| > |z_2| > 0$  and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ .

Next, we consider

$$\begin{aligned} & \langle w'_3, \Omega_+(\mathcal{Y})^0(w_2, z_2) \phi_{g_2^{-1}}(Y_{W_1}^{g_1})^0(u, z_1) w_1 \rangle \\ &= \langle w'_3, \Omega_+(\mathcal{Y})^0(w_2, z_2) (Y_{W_1}^{g_1})^0(g_2 u, z_1) w_1 \rangle \\ &= \langle e^{z_2 L(1)} w'_3, \mathcal{Y}(Y_{W_1}^{g_1}(g_2 u, x) w_1, y) w_2 \Big|_{\substack{x^n = e^{n l_0(z_1)}, \log(x) = l_0(z_1) \\ y^n = e^{n \pi i} e^{n l_0(z_2)}, \log(y) = l_0(z_2) + \pi i}} \rangle \\ &= \langle e^{z_2 L(1)} w'_3, \mathcal{Y}(Y_{W_1}^{g_1}(g_2 u, x) w_1, y) w_2 \Big|_{\substack{x^n = e^{n l_0(z_1)}, \log(x) = l_0(z_1) \\ y^n = e^{n l_{q_2}(-z_2)}, \log(y) = l_{q_2}(-z_2)}} \rangle \\ &= \langle e^{z_2 L(1)} w'_3, \mathcal{Y}^{q_2}((Y_{W_1}^{g_1})^0(g_2 u, z_1) w_1, -z_2) w_2 \rangle, \end{aligned} \quad (4.11)$$

where  $q_2 \in \mathbb{Z}$  is defined in (4.7). From the form of the right-hand side of (4.11) and the duality of  $\mathcal{Y}$ , we know that the right-hand side, therefore the left-hand side, of (4.11) is absolutely convergent on the region given by  $|-z_2| > |z_1| > 0$  (equivalently,  $|z_2| > |z_1| > 0$ ), to a holomorphic function  $h_1(z_1, z_2; u, w_1, w_2, w'_3)$  defined by

$$\left\{ \begin{array}{l} f_{\mathcal{Y},2}^{c^{q_2,0}b_{13}^{-1}}(z_1 - z_2, -z_2; g_2u, w_1, w_2, e^{z_2L(1)}w'_3), \\ \quad \text{if } |z_2| > |z_1| > 0, \quad 3\pi/2 < \arg(z_1 - z_2) - \arg(-z_2) < 2\pi \\ f_{\mathcal{Y},2}^{c^{q_2,0}}(z_1 - z_2, -z_2; g_2u, w_1, w_2, e^{z_2L(1)}w'_3), \\ \quad \text{if } |z_2| > |z_1| > 0, \quad |\arg(z_1 - z_2) - \arg(-z_2)| < \pi/2 \\ f_{\mathcal{Y},2}^{c^{q_2,0}b_{12}^{-1}b_{13}b_{12}}(z_1 - z_2, -z_2; g_2u, w_1, w_2, e^{z_2L(1)}w'_3), \\ \quad \text{if } |z_2| > |z_1| > 0, \quad -2\pi < \arg(z_1 - z_2) - \arg(-z_2) < -3\pi/2. \end{array} \right. \quad (4.12)$$

We need to prove that  $h_1(z_1, z_2; u, w_1, w_2, w'_3) = h_+^e(z_1, z_2; u, w_1, w_2, w'_3)$  on the region given by  $|z_2| > |z_1| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ . Since this region is connected, and both  $h_1$  and  $h_+^e$  are holomorphic on this region, we only need to prove that they are equal in some open subset of this region. So, we consider the nonempty sub-region given by:

$$|z_2| > |z_1| > 0, \quad |z_2 - e^{5\pi i/6}| < 1/10, \quad -2\pi < \arg(z_1 - z_2) - \arg(-z_2) < -\frac{3\pi}{2}.$$

Notice that this is indeed a sub-region, because under the condition  $|z_2 - e^{5\pi i/6}| < 1/10$ , the condition  $-2\pi < \arg(z_1 - z_2) - \arg(-z_2) < -\frac{3\pi}{2}$  implies  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ . In this sub-region, we have  $q_2 = 0$ . So,

$$\begin{aligned} h_1(z_1, z_2; u, w_1, w_2, w'_3) &= f_{\mathcal{Y},2}^{c^{0,0}b_{12}^{-1}b_{13}b_{12}}(z_1 - z_2, -z_2; g_2u, w_1, w_2, e^{z_2L(1)}w'_3) \\ &= f_{\mathcal{Y},2}^{(b_{12}^{-1}b_{13}^{-1}b_{12})b_{12}^{-1}b_{13}b_{12}}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2L(1)}w'_3) \\ &= f_{\mathcal{Y},2}^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2L(1)}w'_3). \end{aligned} \quad (4.13)$$

In the region given by

$$|z_1 - z_2|, |z_1| > |z_2| > 0, \quad |z_2 - e^{5\pi i/6}| < 1/10, \quad |\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}, \quad (4.14)$$

by (4.10) and the fact that  $q_1 = q_2 = 0$ , we have

$$h_+^e(z_1, z_2; u, w_1, w_2, w_3') = f_{\mathcal{Y},2}^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w_3'). \quad (4.15)$$

Since  $h_+^e(z_1, z_2; u, w_1, w_2, w_3')$  is holomorphic on the region given by  $(z_1, z_2) \in F_2(\mathbb{C} - \{0\})$ ,  $z_1, z_2, z_1 - z_2 \notin \mathbb{R}_+$ , and  $f_{\mathcal{Y},2}^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w_3')$  is holomorphic on the region given by  $(z_1 - z_2, -z_2) \in F_2(\mathbb{C} - \{0\})$ ,  $z_1 - z_2, -z_2, (z_1 - z_2) - (-z_2) \notin \mathbb{R}_+$ , we know that they are both holomorphic on the connected open set given by

$$|z_2 - e^{5\pi i/6}| < 1/10, \quad z_1, \quad z_1 - z_2 \notin \mathbb{R}_+, \quad (4.16)$$

which contains the region (4.14) where (4.15) holds. Therefore, we know that (4.15) hold on the region (4.16). Comparing (4.15) and (4.13), we know that

$$h_1(z_1, z_2; u, w_1, w_2, w_3') = h_+^e(z_1, z_2; u, w_1, w_2, w_3')$$

on the region given by  $|z_2| > |z_1| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ .

Next, we consider

$$\begin{aligned} & \langle w_3', \Omega_+(\mathcal{Y})^0((Y_{W_2}^{g_2})^0(u, z_1 - z_2)w_2, z_2)w_1 \rangle \\ &= \langle w_3', \Omega_+(\mathcal{Y})(Y_{W_2}^{g_2}(u, x)w_2, y)w_1 \rangle \Big|_{\substack{x^n = e^{nl_0(z_1 - z_2)}, \log(x) = l_0(z_1 - z_2) \\ y^n = e^{nl_0(z_2)}, \log(y) = l_0(z_2)}} \\ &= \langle e^{z_2 L(1)} w_3', \mathcal{Y}(w_1, y)Y_{W_2}^{g_2}(u, x)w_2 \rangle \Big|_{\substack{x^n = e^{nl_0(z_1 - z_2)}, \log(x) = l_0(z_1 - z_2) \\ y^n = e^{n\pi i} e^{nl_0(z_2)}, \log(y) = l_0(z_2) + \pi i}} \\ &= \langle e^{z_2 L(1)} w_3', \mathcal{Y}(w_1, y)Y_{W_2}^{g_2}(u, x)w_2 \rangle \Big|_{\substack{x^n = e^{nl_0(z_1 - z_2)}, \log(x) = l_0(z_1 - z_2) \\ y^n = e^{nl_{q_2}(-z_2)}, \log(y) = l_{q_2}(-z_2)}} \end{aligned}$$

$$= \langle e^{z_2 L(1)} w'_3, \mathcal{Y}^{q_2}(w_1, -z_2) (Y_{W_2}^{q_2})^0(u, z_1 - z_2) w_2 \rangle, \quad (4.17)$$

where  $q_2 \in \mathbb{Z}$  is defined in (4.7). From the form of the right-hand side of (4.17) and the duality of  $\mathcal{Y}$ , we know that the right-hand side, therefore the left-hand side, of (4.17) is absolutely convergent on the region given by  $|-z_2| > |z_1 - z_2| > 0$  (equivalently,  $|z_2| > |z_1 - z_2| > 0$ ), to a holomorphic function  $h_2(z_1, z_2; u, w_1, w_2, w'_3)$  defined by

$$\begin{cases} f_{\mathcal{Y},2}^{b^{0,q_2}}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3), \\ \quad \text{if } |z_2| > |z_1 - z_2| > 0, \quad -3\pi/2 < \arg z_1 - \arg(-z_2) < -\pi/2 \\ f_{\mathcal{Y},2}^{b^{0,q_2} b_{12}^{-1}}(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3), \\ \quad \text{if } |z_2| > |z_1 - z_2| > 0, \quad \pi/2 < \arg z_1 - \arg(-z_2) < 3\pi/2. \end{cases} \quad (4.18)$$

We need to prove that  $h_2(z_1, z_2; u, w_1, w_2, w'_3)$  and  $h_+^e(z_1, z_2; u, w_1, w_2, w'_3)$  are equal in the connected region given by  $|z_2| > |z_1 - z_2| > 0$ ,  $|\arg z_1 - \arg z_2| < \pi/2$ . Similar to before, we only need to prove this is any open subset of this region. So, we can assume  $|z_2 - e^{5\pi i/6}| < 1/10$ . In this sub-region,  $q_2 = 0$ , and  $\arg(-z_2) = \arg z_2 + \pi$ . Therefore, the condition  $-3\pi/2 < \arg z_1 - \arg(-z_2) < -\pi/2$  is equivalent to  $|\arg z_1 - \arg z_2| < \pi/2$ . So, in this sub-region,

$$h_2(z_1, z_2; u, w_1, w_2, w'_3) = f_{\mathcal{Y},2}^e(z_1 - z_2, -z_2; u, w_1, w_2, e^{z_2 L(1)} w'_3). \quad (4.19)$$

Since this region is a subset of (4.16), we know that (4.15) holds in this region. Thus, on the sub-region, and therefore on the larger region given by  $|z_2| > |z_1 - z_2| > 0$ ,  $|\arg z_1 - \arg z_2| < \pi/2$ , we have

$$h_2(z_1, z_2; u, w_1, w_2, w'_3) = h_+^e(z_1, z_2; u, w_1, w_2, w'_3),$$

which means the duality of  $\Omega_+(\mathcal{Y})$  has been proven.

The  $L(-1)$ -derivative property for  $\Omega_{\pm}(\mathcal{Y})$  comes from (3.51) and the  $L(-1)$ -derivative property for  $\mathcal{Y}$ .

Now we prove the part 3 in the definition of twisted intertwining operator.

$$\begin{aligned}
& \langle w'_3, Y_{W_3}^{g_3}(u_1, z_1) \cdots Y_{W_3}^{g_3}(u_{k-1}, z_{k-1}) \Omega_+(\mathcal{Y})(w_2, z_k) w_1 \rangle \\
&= \langle e^{z_k L(1)} w'_3, Y_{W_3}^{g_3}(u_1, x_1 + y) \cdots Y_{W_3}^{g_3}(u_{k-1}, x_{k-1} + y) \\
& \quad \mathcal{Y}(w_1, y) w_2 \rangle \Big|_{\substack{x_i^n = e^{n \log z_i}, \log x_i = \log z_i, i=1, \dots, k-1, \\ y^n = e^{n \pi i} e^{n l_0(z_k)}, \log y = l_0(z_k) + \pi i}}
\end{aligned} \tag{4.20}$$

Let  $p_1, \dots, p_k$  be integers uniquely determined by  $l_{p_i}(z_i - z_k) = l_0(z_i) - \sum_{l>0} \frac{1}{l} z_k^l z_i^{-l}$ ,  $i = 1, \dots, k-1$ , and  $l_{p_k}(-z_k) = l_0(z_k) + \pi i$ . Then, by Appendix A, (4.20) is absolutely convergent on the region given by  $0 < 2|z_k| < |z_{k-1}| < \dots < |z_1|$ , and is equal to

$$\langle e^{z_k L(1)} w'_3, (Y_{W_3}^{g_3})^{p_1}(u_1, z_1 - z_k) \cdots (Y_{W_3}^{g_3})^{p_{k-1}}(u_{k-1}, z_{k-1} - z_k) \mathcal{Y}^{p_k}(w_1, -z_k) w_2 \rangle. \tag{4.21}$$

On the region  $0 < |z_k| < |z_{k-1} - z_k| < \dots < |z_1 - z_k|$ , (4.21) is equal to a branch of the multi-valued function  $f_{\mathcal{Y},k}(z_1 - z_k, \dots, z_{k-1} - z_k, -z_k)$ , which is defined and analytic on the region  $0 < |z_k| < |z_{k-1}| < \dots < |z_1|$ . Therefore, we know that the region of absolute convergence of the right-hand side, and therefore the left-hand side, of (4.20) can be enlarged to the region given by  $0 < |z_k| < |z_{k-1}| < \dots < |z_1|$ . Let  $f_{\Omega_+(\mathcal{Y}),k}(z_1, \dots, z_k) = f_{\mathcal{Y},k}(z_1 - z_k, \dots, z_{k-1} - z_k, -z_k)$ . By Proposition 3.0.20, we know that for any  $k \in \mathbb{Z}_+$ ,  $f_{\Omega_+(\mathcal{Y}),k}$  satisfies Condition 3.0.13 for  $(k-1, 1, \{0\})$ .  $\square$

## Chapter 5

# Contragredient Isomorphisms $A_{\pm}$

Let  $g_1, g_2$  be automorphisms of  $V$ , and let  $W_1, W_2$  and  $W_3$  be  $g_1$ -,  $g_2$ -, and  $g_1g_2$ -twisted  $V$ -modules. For any linear map  $\mathcal{Y}(\cdot, x) \cdot : W_1 \otimes W_2 \rightarrow W_3\{x\}[\log x]$ , we define  $\mathcal{Y}^{\pm o}$  by

$$\begin{aligned} \mathcal{Y}^{\pm o} : W_1 \otimes W_2 &\rightarrow W_3\{x\}[\log x] \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}^{\pm o}(w_1, x)w_2 = \mathcal{Y}(e^{xL(1)}e^{\pm\pi iL(0)}(x^{-L(0)})^2w_1, x^{-1})w_2. \end{aligned} \quad (5.1)$$

We know that  $\mathcal{Y}^{\pm o}$  is well-defined because  $L(1)_{W_1}$  is locally nilpotent by Assumption 4.0.1. We write

$$\mathcal{Y}^{\pm o}(w_1, x) = \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} (\mathcal{Y}^{\pm o})_{n,k}(w_1) x^{-n-1} (\log x)^k, \quad (\mathcal{Y}^{\pm o})_{n,k}(w_1) \in \text{Hom}_{\mathbb{C}}(W_2, W_3).$$

For the component form, for any  $n \in \mathbb{C}$ ,  $k \in \mathbb{N}$ , and homogeneous  $w_1 \in W_1$ , we have

$$\begin{aligned} (\mathcal{Y}^{\pm o})_{n,k}(w_1) = & \\ \left( \frac{\mp 2}{\pi i} \right)^k e^{\pm\pi i \text{wt } w_1} & \sum_{l \in \mathbb{N}} \sum_{\substack{i, j \in \mathbb{N} \\ i+j \geq k}} \frac{(\pm\pi i)^{i+j}}{2^i j! l!} \binom{j}{k-i} \mathcal{Y}_{-n-l+2 \text{wt } w_1 - 2, i} (L(1)^l L(0)_N^j w_1), \end{aligned} \quad (5.2)$$

which is actually a finite summation.

The powers of  $\log(x)$  in  $\mathcal{Y}^{\pm o}(w_1, x)w_2$  have an upper bound, because of the definition (5.1) and the fact that this holds for  $\mathcal{Y}$ .

We define linear maps

$$\begin{aligned} A_{\pm}(\mathcal{Y}) : W_1 \otimes W'_3 &\rightarrow W'_2\{x\}[\log x] \\ w_1 \otimes w'_3 &\mapsto A_{\pm}(\mathcal{Y})(w_1, x)w'_3 \end{aligned}$$

by

$$\langle A_{\pm}(\mathcal{Y})(w_1, x)w'_3, w_2 \rangle = \langle w'_3, \mathcal{Y}^{\pm o}(w_1, x)w_2 \rangle \quad (5.3)$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$  and  $w'_3 \in W'_3$ .

The powers of  $\log(x)$  in  $A_{\pm}(\mathcal{Y})(w_1, x)w'_3$  have an upper bound, because of the definition (5.3), (5.1), Assumption 4.0.1, and Remark 4.0.3.

Let  $(W, Y_W^g)$  be a  $g$ -twisted  $V$ -module. When  $W_1 = V$ ,  $W_2 = W_3 = W$  and  $\mathcal{Y} = Y_W^g$ , by definition,  $A_+(Y_W^g) = A_-(Y_W^g) = (Y_W^g)'$  (see Section 3).

Let  $L_{W_1}(0)_S$  be the semisimple part of  $L_{W_1}(0)$ . From the definition (5.3), for  $p \in \mathbb{Z}$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w'_3 \in W'_3$  and  $z \in \mathbb{C}^\times$ , we have

$$\begin{aligned} &\langle A_{\pm}(\mathcal{Y})^p(w_1, z)w'_3, w_2 \rangle \\ &= \langle A_{\pm}(\mathcal{Y})^p(w_1, x)w'_3, w_2 \rangle \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)} \\ &= \langle w'_3, \mathcal{Y}(e^{xL_{W_1}(1)}e^{\pm\pi iL_{W_1}(0)}(x^{-L_{W_1}(0)})^2w_1, x^{-1})w_2 \rangle \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)} \\ &= \langle w'_3, \mathcal{Y}(e^{xL_{W_1}(1)}e^{\pm\pi iL_{W_1}(0)}(x^{-L_{W_1}(0)_S})^2 \\ &\quad \cdot (e^{-(L_{W_1}(0)-L_{W_1}(0)_S)\log x})^2w_1, x^{-1})w_2 \rangle \Big|_{x^n=e^{nl_p(z)}, \log x=l_p(z)} \\ &= \langle w'_3, \mathcal{Y}(e^{zL_{W_1}(1)}e^{\pm\pi iL_{W_1}(0)}(e^{-l_p(z)L_{W_1}(0)_S})^2 \end{aligned}$$

$$\begin{aligned}
& \cdot \langle (e^{-(L_{W_1(0)} - L_{W_1(0)S})l_p(z)})^2 w_1, y \rangle w_2 \Big|_{y^n = e^{-nl_p(z)}, \log y = -l_p(z)} \\
& = \langle w'_3, \mathcal{Y}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{-2l_p(z)L_{W_1(0)}} w_1, y) w_2 \rangle \Big|_{y^n = e^{-nl_p(z)}, \log y = -l_p(z)}. \quad (5.4)
\end{aligned}$$

When  $\arg z = 0$ ,  $\arg z^{-1} = \arg z = 0$  and  $-l_p(z) = l_{-p}(z^{-1})$ . When  $\arg z \neq 0$ ,  $\arg z^{-1} = -\arg z + 2\pi$  and  $-l_p(z) = l_{-p-1}(z^{-1})$ . Hence when  $\arg z = 0$ , the right-hand side of (5.4) is equal to

$$\begin{aligned}
& \langle w'_3, \mathcal{Y}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{2l_{-p}(z^{-1})L_{W_1(0)}} w_1, y) w_2 \rangle \Big|_{y^n = e^{nl_{-p}(z^{-1})}, \log y = l_{-p}(z^{-1})} \\
& = \langle w'_3, \mathcal{Y}^{-p}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{2l_{-p}(z^{-1})L_{W_1(0)}} w_1, z^{-1}) w_2 \rangle \quad (5.5)
\end{aligned}$$

and when  $\arg z \neq 0$ , it is equal to

$$\begin{aligned}
& \langle w'_3, \mathcal{Y}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{2l_{-p-1}(z^{-1})L_{W_1(0)}} w_1, y) w_2 \rangle \Big|_{y^n = e^{nl_{-p-1}(z^{-1})}, \log y = l_{-p-1}(z^{-1})} \\
& = \langle w'_3, \mathcal{Y}^{-p-1}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{2l_{-p-1}(z^{-1})L_{W_1(0)}} w_1, z^{-1}) w_2 \rangle. \quad (5.6)
\end{aligned}$$

From (5.4)–(5.6), for  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w'_3 \in W'_3$  and  $z \in \mathbb{C}^\times$ , we have

$$\langle A_\pm(\mathcal{Y})^p(w_1, z) w'_3, w_2 \rangle = \langle w'_3, \mathcal{Y}^{-p}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{2l_{-p}(z^{-1})L_{W_1(0)}} w_1, z^{-1}) w_2 \rangle \quad (5.7)$$

when  $\arg z = 0$  and

$$\langle A_\pm(\mathcal{Y})^p(w_1, z) w'_3, w_2 \rangle = \langle w'_3, \mathcal{Y}^{-p-1}(e^{zL_{W_1(1)}} e^{\pm\pi i L_{W_1(0)}} e^{2l_{-p-1}(z^{-1})L_{W_1(0)}} w_1, z^{-1}) w_2 \rangle \quad (5.8)$$

when  $\arg z \neq 0$ .

**Theorem 5.0.1.** *The linear maps  $A_+(\mathcal{Y})$  and  $A_-(\mathcal{Y})$  are twisted intertwining operators of types  $(\phi_{g_1}(W'_3))$  and  $(W_1 \phi_{g_1^{-1}}(W'_3))$ , respectively. If  $\mathcal{Y}$  is fully lower truncated, then  $A_\pm(\mathcal{Y})$  is also fully lower truncated.*

*Proof.* The second sentence in this theorem is clear by the definition of  $A_{\pm}(\mathcal{Y})$ . Now we prove the first sentence.

We shall start with the duality. Let  $u \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w_3 \in W'_3$ . For  $i = 1, 2$ , define  $p_i = p_i(z_i) = -1$  if  $\arg z_i \neq 0$ , and 0 if  $\arg z_i = 0$ . We know that  $l_{p_i}(z_i^{-1}) = -l_0(z_i)$ . First, we consider

$$\begin{aligned}
& \langle \phi_{g_1}((Y_{W_2}^{g_2})')^0(u, z_1)A_+(\mathcal{Y})^0(w_1, z_2)w'_3, w_2 \rangle \\
&= \langle ((Y_{W_2}^{g_2})')^0(g_1^{-1}u, z_1)A_+(\mathcal{Y})^0(w_1, z_2)w'_3, w_2 \rangle \\
&= \left\langle w'_3, \mathcal{Y}^{p_2}(e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)} w_1, z_2^{-1}) \right. \\
&\quad \left. \cdot (Y_{W_2}^{g_2})^{p_1} \left( e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1}u, z_1^{-1} \right) w_2 \right\rangle \quad (5.9)
\end{aligned}$$

Since  $e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)} w_1$  and  $e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1}u$  are Laurent polynomials in  $z_2$  and  $z_1$  respectively, the right-hand side of (5.9) is a finite sum of series of form (3.23). Because of the duality of  $\mathcal{Y}$  and the form of the right-hand side of (5.9), the series (5.9) is absolutely convergent, in the region given by  $|z_2^{-1}| > |z_1^{-1}| > 0$  (or equivalently,  $|z_1| > |z_2| > 0$ ), to the function

$$\left\{ \begin{array}{l} f_{\mathcal{Y},2}^{b_{p_1}, p_2}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1}u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)} w_1, w_2, w'_3), \\ \quad \text{if } |z_2^{-1}| > |z_1^{-1}| > 0, \quad -\frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < -\frac{\pi}{2} \\ f_{\mathcal{Y},2}^{b_{p_1}, p_2, b_{12}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1}u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)} w_1, w_2, w'_3), \\ \quad \text{if } |z_2^{-1}| > |z_1^{-1}| > 0, \quad \frac{\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < \frac{3\pi}{2} \end{array} \right. , \quad (5.10)$$

which is holomorphic on the region given by  $|z_2^{-1}| > |z_1^{-1}| > 0$ , and  $z_1^{-1}, z_2^{-1} \notin \mathbb{R}_+$  (or equivalently,  $|z_1| > |z_2| > 0$ , and  $z_1, z_2 \notin \mathbb{R}_+$ ). Also, because of the duality of  $\mathcal{Y}$  and the form of the right-hand side of (5.9), the holomorphic function (5.10) can be analytically extended to a multivalued holomorphic function from the region given by  $|z_1| > |z_2| > 0$  to the region given by  $z_1^{-1}, z_2^{-1}, z_1^{-1} - z_2^{-1} \neq 0$ , which is equivalent

to  $z_1, z_2, z_1 - z_2 \neq 0$ . Let  $h_+(z_1, z_2; u, w_1, w_2, w'_3)$  be the multivalued holomorphic function on  $F_2(\mathbb{C} - \{0\})$  with a preferred single-valued branch on the region given by cutting lines  $z_1 \in \mathbb{R}_+$ ,  $z_2 \in \mathbb{R}_+$  and  $z_1 - z_2 \in \mathbb{R}_+$  in  $F_2(\mathbb{C} - \{0\})$ , such that  $h_+^e(z_1, z_2; u, w_1, w_2, w'_3)$  is equal to (5.10) when  $|z_1| > |z_2| > 0$ , and  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ .

Then, we need to consider

$$\begin{aligned} & \langle A_+(\mathcal{Y})^0(w_1, z_2)((Y_{W_3}^{g_3})')^0(u, z_1)w'_3, w_2 \rangle \\ &= \left\langle w'_3, (Y_{W_3}^{g_3})^{p_1} \left( e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} u, z_1^{-1} \right) \cdot \right. \\ & \quad \left. \mathcal{Y}^{p_2} (e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, z_2^{-1}) w_2 \right\rangle. \end{aligned} \quad (5.11)$$

Because of the form of the right-hand side of (5.11), the duality of  $\mathcal{Y}$ , and lemma 3.0.33, we know that (5.11) is absolutely convergent, in the region given by  $|z_1^{-1}| > |z_2^{-1}| > 0$  (or equivalently,  $|z_2| > |z_1| > 0$ ), to a holomorphic function  $h_1(z_1, z_2; u, w_1, w_2, w'_3)$  defined by

$$\left\{ \begin{array}{l} f_{\mathcal{Y},2}^{a^{p_1 \cdot p_2}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, w_2, w'_3), \\ \quad \text{if } |z_1^{-1}| > |z_2^{-1}| > 0, \quad |\arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1}| < \frac{\pi}{2} \\ f_{\mathcal{Y},2}^{a^{p_1 \cdot p_2} b_{12}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, w_2, w'_3), \\ \quad \text{if } |z_1^{-1}| > |z_2^{-1}| > 0, \quad -2\pi < \arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1} < -\frac{3\pi}{2} \\ f_{\mathcal{Y},2}^{a^{p_1 \cdot p_2} b_{12}^{-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, w_2, w'_3), \\ \quad \text{if } |z_1^{-1}| > |z_2^{-1}| > 0, \quad \frac{3\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1} < 2\pi \end{array} \right. \quad (5.12)$$

We need to prove that  $h_1(z_1, z_2; u, w_1, w_2, w'_3) = h_+^e(z_1, z_2; u, w_1, w_2, w'_3)$  in the region given by  $|z_2| > |z_1| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ . Since this region is connected, and both  $h_1$  and  $h_+^e$  are holomorphic on this region, we only need to prove

that they are equal in any open subset. Let  $U = \{z \mid |z - (-0.6 + 0.8i)| < 1/10\}$ . We consider the following open subset

$$\mathcal{R}_1 = \left\{ (z_1, z_2) \in M^2 \mid z_2 \in U, |z_2| > |z_1| > 0, -\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2} \right\}.$$

Notice that if  $z_2 \in U$ ,  $|z_2| > |z_1| > 0$ , then

$$\arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1} = (\arg(z_1 - z_2) - \arg z_2) + \pi. \quad (5.13)$$

Therefore, we know that

$$\mathcal{R}_1 \subset \left\{ (z_1, z_2) \in M^2 \mid |z_2| > |z_1| > 0, |\arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1}| < \frac{\pi}{2} \right\}.$$

If  $(z_1, z_2) \in \mathcal{R}_1$ , we have  $p_1(z_1) = p_2(z_2) = -1$ . By (5.12), for any  $(z_1, z_2) \in \mathcal{R}_1$ , we have

$$\begin{aligned} h_1(z_1, z_2; u, w_1, w_2, w'_3) &= f^{(b_{13}b_{12})^{-1}b_{23}^{-1}}(z_1^{-1}, z_2^{-1}; \\ &e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{(\pi i + 2l_{-1}(z_2^{-1}))L_{W_1}(0)}w_1, w_2, w'_3). \end{aligned} \quad (5.14)$$

On the other hand, notice that

$$\begin{aligned} \mathcal{R}_2 &:= \left\{ (z_1, z_2) \in M^2 \mid z_2 \in U, |z_1| > |z_2| > 0, \frac{\pi}{2} < \arg(z_1^{-1} - z_2^{-1}) - \arg z_2^{-1} < \frac{3\pi}{2} \right\} \\ &\subset \left\{ (z_1, z_2) \in M^2 \mid |z_1| > |z_2| > 0, |\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2} \right\}. \end{aligned} \quad (5.15)$$

We know that  $h_+^e$  is equal to (5.10) when  $|z_1| > |z_2| > 0$ ,  $|\arg(z_1 - z_2) - \arg z_1| < \frac{\pi}{2}$ , which is the right-hand side of (5.15). Also, if  $(z_1, z_2) \in \mathcal{R}_2$ , then  $p_1(z_1) = p_2(z_2) =$

–1. Therefore, for any  $(z_1, z_2) \in \mathcal{R}_2$ , we know that

$$\begin{aligned}
& h_+^e(z_1, z_2; u, w_1, w_2, w_3') \\
&= f^{(b_{12}^{-1}b_{13}b_{12})^{-1}(b_{23}b_{12})^{-1}b_{12}^{-1}} \left( z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1} u, \right. \\
&\quad \left. e^{z_2 L_{W_1}(1)} e^{(\pi l + 2l - p_2 - 1)(z_2^{-1})} L_{W_1(0)} w_1, w_2, w_3' \right) \\
&= f^{b_{12}(b_{12}^{-1}b_{13}b_{12})^{-1}(b_{23}b_{12})^{-1}b_{12}^{-1}} \left( z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} u, \right. \\
&\quad \left. e^{z_2 L_{W_1}(1)} e^{(\pi l + 2l - p_2 - 1)(z_2^{-1})} L_{W_1(0)} w_1, w_2, w_3' \right) \\
&= f^{(b_{13}b_{12})^{-1}b_{23}^{-1}} \left( z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} u, \right. \\
&\quad \left. e^{z_2 L_{W_1}(1)} e^{(\pi l + 2l - p_2 - 1)(z_2^{-1})} L_{W_1(0)} w_1, w_2, w_3' \right) \quad (5.16)
\end{aligned}$$

Notice that  $\mathcal{R}_1 \cup \mathcal{R}_2$  is contained in the simply-connected region  $\Omega$  defined by

$$\begin{aligned}
& \left\{ (z_1, z_2) \in M^2 \mid z_1, z_2, (z_1 - z_2), (z_1^{-1} - z_2^{-1}) \notin \mathbb{R}_{\geq 0} \right\} \\
& - \left\{ (z_1, z_2) \in M^2 \mid |z_1| > |z_2| > 0, \frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_1 < 2\pi \right\} \\
& - \left\{ (z_1, z_2) \in M^2 \mid |z_2| > |z_1| > 0, -2\pi < \arg(z_1^{-1} - z_2^{-1}) - \arg z_1^{-1} < -\frac{3\pi}{2} \right\}.
\end{aligned}$$

Since both of the left-hand side and the right-hand side of (5.16) are holomorphic in  $\Omega$ , and they are equal in  $\mathcal{R}_2$ , we know that they are equal in  $\Omega$ . In particular, they are equal in  $\mathcal{R}_1$ . In other words, (5.16) holds for  $(z_1, z_2) \in \mathcal{R}_1$ . Therefore, comparing with (5.16) and (5.14), we know that on the region  $\mathcal{R}_1$ , and therefore on the larger region given by  $|z_2| > |z_1| > 0, -\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ ,

$$h_1(z_1, z_2; u, w_1, w_2, w_3') = h_+^e(z_1, z_2; u, w_1, w_2, w_3').$$

Next, we consider

$$\langle A_{\pm}(\mathcal{Y})^0 ((Y_{W_1}^{g_1})^0(u, z_1 - z_2) w_1, z_2) w_3', w_2 \rangle$$

$$\begin{aligned}
& = \langle A_{\pm}(\mathcal{Y}) (Y_{W_1}^{g_1}(u, x)w_1, y) w'_3, w_2 \rangle \Big|_{\substack{x^n=e^{n l_0(z_1-z_2)}, \log(x)=l_0(z_1-z_2) \\ y^n=e^{n l_0(z_2)}, \log(y)=l_0(z_2)}} \\
& = \langle w'_3, \mathcal{Y} (e^{yL(1)} e^{\pm\pi i L(0)} (y^{-L(0)})^2 Y_{W_1}^{g_1}(u, x)w_1, y^{-1}) w_2 \rangle \Big|_{\substack{x^n=e^{n l_0(z_1-z_2)}, \log(x)=l_0(z_1-z_2) \\ y^n=e^{n l_0(z_2)}, \log(y)=l_0(z_2)}}
\end{aligned} \tag{5.17}$$

Same as the formula (6.30) in [H9], we have

$$\begin{aligned}
& e^{yL(1)} e^{\pm\pi i L(0)} (y^{-L(0)})^2 Y_{W_1}^{g_1}(u, x) \\
& = Y_{W_1}^{g_1} \left( e^{(y+x)L_V(1)} (-(y+x)^{-2})^{L_V(0)} u, \frac{e^{\pm\pi i x}}{(y+x)y} \right) e^{yL_{W_1}(1)} e^{\pm\pi i L_{W_1}(0)} (y^{-L_{W_1}(0)})^2
\end{aligned}$$

Therefore, the right-hand side of (5.17) is equal to

$$\begin{aligned}
& \left\langle w'_3, \mathcal{Y} \left( Y_{W_1}^{g_1} \left( e^{(y+x)L_V(1)} (-(y+x)^{-2})^{L_V(0)} u, \frac{\tilde{x}x}{(y+x)y} \right) e^{yL_{W_1}(1)} \right. \right. \\
& \quad \left. \left. e^{\pm\pi i L_{W_1}(0)} (y^{-L_{W_1}(0)})^2 w_1, y^{-1} \right) w_2 \right\rangle \Big|_{\substack{x^n=e^{n l_0(z_1-z_2)}, \log(x)=l_0(z_1-z_2) \\ y^n=e^{n l_0(z_2)}, \log(y)=l_0(z_2) \\ \tilde{x}^n=e^{\pm n \pi i}, \log(\tilde{x})=\pm\pi i}}
\end{aligned} \tag{5.18}$$

We consider

$$\begin{aligned}
& \left\langle w'_3, \mathcal{Y}^{p_2} \left( (Y_{W_1}^{g_1})^{N_{\pm}} (e^{z_1 L(1)} (-z_1^{-2})^{L(0)} u, z_1^{-1} - z_2^{-1}) \right. \right. \\
& \quad \left. \left. e^{z_2 L_{W_1}(1)} e^{(\pm\pi i + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)} w_1, z_2^{-1} \right) w_2 \right\rangle, \tag{5.19}
\end{aligned}$$

where  $N_{\pm} = N_{\pm}(z_1, z_2)$  is the unique integer such that for any  $\alpha \in \mathbb{C}$ ,

$$\begin{aligned}
& l_{N_{\pm}}(z_1^{-1} - z_2^{-1}) = \left[ \log \left( \frac{\tilde{x}x}{(y+x)y} \right) \right] \Big|_{\substack{x^n=e^{n l_0(z_1-z_2)}, \log(x)=l_0(z_1-z_2) \\ y^n=e^{n l_0(z_2)}, \log(y)=l_0(z_2) \\ \tilde{x}^n=e^{\pm n \pi i}, \log(\tilde{x})=\pm\pi i}} \\
& e^{\alpha l_{N_{\pm}}(z_1^{-1} - z_2^{-1})} = \left( \frac{\tilde{x}x}{(y+x)y} \right)^{\alpha} \Big|_{\substack{x^n=e^{n l_0(z_1-z_2)}, \log(x)=l_0(z_1-z_2) \\ y^n=e^{n l_0(z_2)}, \log(y)=l_0(z_2) \\ \tilde{x}^n=e^{\pm n \pi i}, \log(\tilde{x})=\pm\pi i}}
\end{aligned} \tag{5.20}$$

Actually, (5.20) is equivalent to

$$l_{N_{\pm}}(z_1^{-1} - z_2^{-1}) = \pm\pi\iota + l_0(z_1 - z_2) - 2l_0(z_2) + \sum_{k>0} \frac{(-1)^k}{k} z_2^{-k} (z_1 - z_2)^k.$$

Similar to the argument for proving  $\langle w'_3, (Y_{W_3}^{g_3})^0(u, z_1)\Omega_+(\mathcal{Y})^0(w_2, z_2)w_1 \rangle$  is absolutely convergent on the region given by  $|z_1|/2 > |z_2| > 0$ , we can show that the left-hand side of (5.17) is absolutely convergent to (5.19) on the region given by  $|z_2|/2 > |z_1 - z_2| > 0$ . By the duality of  $\mathcal{Y}$ , we know that (5.19) is absolutely convergent, on the region given by  $|z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$ , to

$$\left\{ \begin{array}{l} f^{c^{p_2, N_{\pm}}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{(\pm\pi\iota + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)}w_1, w_2, w'_3), \\ \quad \text{if } |z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0, \quad |\arg(z_1^{-1}) - \arg(z_2^{-1})| < \frac{\pi}{2} \\ f^{c^{p_2, N_{\pm}b_{12}^{-1}b_{13}b_{12}}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{(\pm\pi\iota + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)}w_1, w_2, w'_3), \\ \quad \text{if } |z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0, \quad -2\pi < \arg(z_1^{-1}) - \arg(z_2^{-1}) < -\frac{3\pi}{2} \\ f^{c^{p_2, N_{\pm}b_{13}^{-1}}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)}u, e^{z_2 L_{W_1}(1)}e^{(\pm\pi\iota + 2l_{p_2}(z_2^{-1}))L_{W_1}(0)}w_1, w_2, w'_3), \\ \quad \text{if } |z_2^{-1}| > |z_1^{-1} - z_2^{-1}| > 0, \quad \frac{3\pi}{2} < \arg(z_1^{-1}) - \arg(z_2^{-1}) < 2\pi \end{array} \right. \quad (5.21)$$

By the duality of  $\mathcal{Y}$ , (5.19), therefore (5.17) can be analytically extended to a multi-valued holomorphic function whose singularities are only  $z_1^{-1}, z_2^{-1}, z_1^{-1} - z_2^{-1} = 0$ , or equivalently  $z_1, z_2, z_1 - z_2 \neq 0$ . So, the left-hand side of (5.17) can be analytically extended from  $|z_2|/2 > |z_1 - z_2| > 0$  to  $|z_2| > |z_1 - z_2| > 0$ . The left-hand side of (5.17) is a finite sum of multi-valued function of the form  $e^{\alpha l_0(z_1 - z_2)} e^{\beta l_0(z_2)} l_0(z_1 - z_2)^i l_0(z_2)^j$  ( $\alpha, \beta \in \mathbb{C}, i, j \in \mathbb{N}$ ) multiplying power series on  $\frac{z_1 - z_2}{z_2}$ . The fact that (5.17) can be analytically extended from  $|z_2|/2 > |z_1 - z_2| > 0$  to  $|z_2| > |z_1 - z_2| > 0$  implies the radius of convergence of each power series is not less than 1. Thus, as a series, (5.17) is absolutely convergent to (5.21) on the region given by  $|z_2| > |z_1 - z_2| > 0$ .

Let  $h_2(z_1, z_2; u, w_1, w_2, w'_3)$  be the holomorphic function on the region given by

$(z_1, z_2) \in M^2$ ,  $z_1, z_2, z_1 - z_2 \notin \mathbb{R}$ , such that  $h_2(z_1, z_2; u, w_1, w_2, w'_3)$  is equal to (5.17) on the region given by  $|z_2| > |z_1 - z_2| > 0$ , and  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ .

When  $|z_2 - e^{\frac{5\pi i}{6}}| < 1/10$ , and  $|z_2| > |z_1 - z_2| > 0$ , we have  $\arg z_1, \arg z_2 \neq 0$ , which implies  $\arg(z_i^{-1}) = 2\pi - \arg z_i$ . So, we know that in the region given by

$$\begin{aligned} &|z_2 - e^{\frac{5\pi i}{6}}| < 1/10, |z_2| > |z_1 - z_2| > 0, |z_2^{-1}| > |z_1^{-1} - z_2^{-1}|, \\ &|\arg z_1 - \arg z_2| < \frac{\pi}{2}, \quad -\frac{3\pi}{2} \arg(z_1^{-1} - z_2^{-1}) - \arg(z_2^{-1}) < -\frac{\pi}{2}, \\ &\arg z_1 \neq 0, \arg z_2 \neq 0 \end{aligned} \tag{5.22}$$

$h_2(z_1, z_2; u, w_1, w_2, w'_3)$ , and therefore (5.17), is equal to

$$f^{c^{-1,0}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)} u, e^{z_2 L_{W_1}(1)} e^{(\pm\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, w_2, w'_3). \tag{5.23}$$

Notice that actually the first line of (A.27) implies the second and third lines of (5.22).

Also, in the region (A.27), we know  $N_+ = 0$ , and  $p_2 = -1$ .

On the other hand, in the same region,  $h_+^e(z_1, z_2; u, w_1, w_2, w'_3)$  is equal to

$$\begin{aligned} &f^{b^{-1,-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)} g_1^{-1} u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, w_2, w'_3) \\ &= f^{b_{12} b^{-1,-1}}(z_1^{-1}, z_2^{-1}; e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)} u, e^{z_2 L_{W_1}(1)} e^{(\pi i + 2l_{p_2}(z_2^{-1})) L_{W_1}(0)} w_1, w_2, w'_3) \end{aligned} \tag{5.24}$$

Since  $b_{12} b^{-1,-1} = c^{-1,0}$ , we know that on the region (5.22), and therefore on the larger region given by  $|z_2| > |z_1 - z_2| > 0$ ,  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ ,

$$h_2(z_1, z_2; u, w_1, w_2, w'_3) = h_+^e(z_1, z_2; u, w_1, w_2, w'_3).$$

Using the same argument for proving the  $L(-1)$ -derivative property for  $\Omega_+(\mathcal{Y})$ , we can show that the  $L(-1)$ -derivative property holds for  $A_+(\mathcal{Y})$ .

Now we prove the part 3 in the definition of twisted intertwining operator. By the definition of  $A_+$ , we have

$$\begin{aligned}
& \langle \phi_{g_1}((Y_{W_2}^{g_2})')(u_1, z_1) \cdots \phi_{g_1}((Y_{W_2}^{g_2})')(u_{k-1}, z_{k-1}) A_+(\mathcal{Y})(w_1, z_k) w'_3, w_2 \rangle \\
&= \langle w'_3, \mathcal{Y}^{p_k} (e^{z_k L_{W_1}(1)} e^{\pi i L_{W_1}(0)} e^{-2 \log(z_k) L_{W_1}(0)} w_1, z_k^{-1}) \cdot \\
&\quad \cdot (Y_{W_2}^{g_2})^{p_{k-1}} (e^{z_{k-1} L_V(1)} (-z_{k-1}^{-2})^{L_V(0)} g_1^{-1} u_{k-1}, z_{k-1}^{-1}) \cdot \\
&\quad \cdots (Y_{W_2}^{g_2})^{p_1} (e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1} u_1, z_1^{-1}) w_2 \rangle, \tag{5.25}
\end{aligned}$$

where  $p_i \in \mathbb{Z}$  is uniquely determined by  $l_{p_i}(z_i^{-1}) = -l_0(z_i)$ . By Lemma 3.0.37, we know that on the region given by  $0 < |z_1^{-1}| < \cdots < |z_k^{-1}|$ , or equivalently,  $|z_1| > \cdots > |z_k| > 0$ , (5.25) is absolutely convergent to

$$\begin{aligned}
& f_{\mathcal{Y}, k} \left( z_{k-1}^{-1}, \dots, z_1^{-1}, z_k^{-1}; e^{z_{k-1} L_V(1)} (-z_{k-1}^{-2})^{L_V(0)} g_1^{-1} u_{k-1}, \dots, e^{z_1 L_V(1)} (-z_1^{-2})^{L_V(0)} g_1^{-1} u_1, \right. \\
& \left. e^{z_k L_{W_1}(1)} e^{\pi i L_{W_1}(0)} e^{-2 \log(z_k) L_{W_1}(0)} w_1, w_2, w'_3 \right) \tag{5.26}
\end{aligned}$$

Let  $f_{A_+(\mathcal{Y}), k}(z_1, \dots, z_k; u_1, \dots, u_{n-1}, w_1, w'_3, w_2)$  be equal to (5.26). By Proposition 3.0.20, we know that for any  $k \in \mathbb{Z}_+$ , the function  $f_{A_+(\mathcal{Y}), k}$  satisfies Condition 3.0.13 for  $(k-1, 1, \{0\})$ .  $\square$

## Chapter 6

# Tensor Product Bifunctors

In this section we introduce the notion of twisted  $P(z)$ -intertwining map and give a definition of tensor product of two twisted modules. The material in this section is essentially the same as the corresponding material in [HL2] and [HLZ3] except that  $V$ -modules and intertwining maps are replaced by twisted modules and twisted intertwining maps.

Let  $G$  be a group of automorphisms of  $V$  and  $\mathcal{C}$  a category of  $g$ -twisted  $V$ -modules for  $g \in G$ . Let  $\mathcal{C}_g$  be the full subcategory of  $\mathcal{C}$  whose objects are the  $g$ -twisted objects in  $\mathcal{C}$ . The category  $\mathcal{C}$  can be the category  $\mathcal{GM}_{gr}(G)$  (the category of grading restricted generalized  $g$ -twisted  $V$ -modules for  $g \in G$ ) or  $\mathcal{M}_{gr}(G)$  (the category of grading restricted ordinary  $g$ -twisted  $V$ -modules for  $g \in G$ ). But since many of the constructions in the present paper works for any category satisfying suitable conditions, we shall work with a general category  $\mathcal{C}$ .

**Definition 6.0.1.** Let  $g_1$  and  $g_2$  be automorphisms of  $V$ , and let  $W_1, W_2, W_3$  be  $g_1$ -,  $g_2$ -,  $g_1g_2$ -twisted  $V$ -modules, respectively, and let  $z \in \mathbb{C}^\times$ . A *twisted  $P(z)$ -intertwining map of type  $\binom{W_3}{W_1W_2}$*  is a linear map  $I : W_1 \otimes W_2 \rightarrow \overline{W_3}$  given by  $I(w_1 \otimes w_2) = \mathcal{Y}(w_1, z)w_2$  for  $w_1 \in W_1$  and  $w_2 \in W_2$ , where  $\mathcal{Y}$  is a twisted intertwining operator of type  $\binom{W_3}{W_1W_2}$ .

Using Definition 3.0.23 and the  $L(0)$ -conjugation formula we can prove the following proposition.

**Proposition 6.0.2.** *Let  $I$  be a linear map from  $W_1 \otimes W_2$  to  $\overline{W_3}$ , and  $z \in \mathbb{C}^\times$ . Then,  $I$  is a  $P(z)$ -intertwining map of type  $\binom{W_3}{W_1 W_2}$  if and only if*

$$\mathcal{Y}_I(w_1, x)w_2 = x^{L(0)}z^{-L(0)}I(x^{-L(0)}z^{L(0)}w_1 \otimes x^{-L(0)}z^{L(0)}w_2) \in W_3\{x\}[\log(x)] \quad (6.1)$$

is a twisted intertwining operator of type  $\binom{W_3}{W_1 W_2}$ , where

$$z^{\pm L(0)} = e^{\pm l_0(z)L(0)_S} \sum_{k \in \mathbb{N}} \frac{(\pm l_0(z))^k}{k!} L(0)_N^k.$$

Using the notion of twisted  $P(z)$ -intertwining map, we can define the notion of tensor product of two twisted modules in  $\mathcal{C}$ .

**Definition 6.0.3.** Let  $W_1$  and  $W_2$  be  $g_1$ - and  $g_2$ -twisted  $V$ -modules, respectively. A  $P(z)$ -product of  $W_1$  and  $W_2$  is a pair  $(W_3, I)$  consisting of a  $g_1 g_2$ -twisted  $V$ -module  $W_3$  and a twisted  $P(z)$ -intertwining map  $I$  of type  $\binom{W_3}{W_1 W_2}$ . If  $W_1$  and  $W_2$  are objects in  $\mathcal{C}$ , a  $P(z)$ -tensor product of  $W_1$  and  $W_2$  in  $\mathcal{C}$  is a  $P(z)$ -product  $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$  satisfying the following universal property:

1. The module  $W_1 \boxtimes_{P(z)} W_2$  is an object in  $\mathcal{C}$ .
2. For any  $P(z)$ -product  $(W_3, I)$  of  $W_1$  and  $W_2$  with  $W_3 \in \text{ob } \mathcal{C}$ , there exists a unique module map  $f : W_1 \boxtimes_{P(z)} W_2 \rightarrow W_3$  such that we have the commutative diagram

$$\begin{array}{ccc} W_1 \otimes W_2 & \xrightarrow{I} & \overline{W_3} \\ \boxtimes_{P(z)} \downarrow & \nearrow f & \\ \overline{W_1 \boxtimes_{P(z)} W_2} & & \end{array}$$

where  $\bar{f}$  is the natural extension of  $f$  to  $\overline{W_1 \boxtimes_{P(z)} W_2}$ .

We now give a construction of the  $P(z)$ -tensor product  $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$  of  $W_1$  and  $W_2$  in  $\mathcal{C}$  under a suitable assumption using the same method as in [HL2] and [HLZ3].

Given a  $P(z)$ -product  $(W_3, I)$  of  $W_1$  and  $W_2$ , for  $w'_3 \in W'_3$ , we have an element  $\lambda_{I, w'_3} \in (W_1 \otimes W_2)^*$  defined by

$$\lambda_{I, w'_3}(w_1 \otimes w_2) = \langle w'_3, I(w_1 \otimes w_2) \rangle$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . Let  $W_1 \boxtimes_{P(z)} W_2$  be the subspace of  $(W_1 \otimes W_2)^*$  spanned by  $\lambda_{I, w'_3}$  for all  $P(z)$ -products  $(W_3, I)$  and  $w'_3 \in W'_3$ , with  $W_3$  being a  $g_1 g_2$ -twisted object in  $\mathcal{C}$ . It is important to keep in mind that the space  $W_1 \boxtimes_{P(z)} W_2$  depends on  $\mathcal{C}$ . We define a vertex operator map

$$Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1 g_2)^{-1}} : V \otimes (W_1 \boxtimes_{P(z)} W_2) \rightarrow (W_1 \boxtimes_{P(z)} W_2)\{x\}[\log x]$$

by

$$Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1 g_2)^{-1}}(v, x)\lambda_{I, w'_3} = \lambda_{I, Y_{W'_3}^{(g_1 g_2)^{-1}}(v, x)w'_3} \quad (6.2)$$

for  $v \in V$  and  $\lambda_{I, w'_3} \in W_1 \boxtimes_{P(z)} W_2$ .

**Proposition 6.0.4.**  $Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1 g_2)^{-1}}(\cdot, x)\cdot$  is a well-defined linear map.

*Proof.* Suppose  $J$  is a finite index set, and  $\lambda = \sum_{i \in J} \lambda_{I_i, w'_{3,i}}$ , where  $w'_{3,i} \in W'_{3,i}$ ,  $W_{3,i}$  is a  $g_1 g_2$ -twisted module in  $\mathcal{C}$ , and  $I_i(\cdot \otimes \cdot) = \mathcal{Y}_i(\cdot, z)\cdot$  is a twisted intertwining map of type  $\left(\begin{smallmatrix} W_{3,i} \\ W_1 W_2 \end{smallmatrix}\right)$ , for  $i \in J$ . We need to prove that if  $\lambda = 0$ , then  $Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1 g_2)^{-1}}(v, x)(\lambda) = 0$ , or equivalently,  $(Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1 g_2)^{-1}})^o(v, x)(\lambda) = 0$ , for any  $v \in V$ .

Now assume  $\lambda = 0$ . For any  $w_1 \in W_1$ ,  $w_2 \in W_2$ , we know that

$$0 = \lambda(w_1 \otimes w_2) = \lambda = \sum_{i \in J} \lambda_{I_i, w'_{3,i}}(w_1 \otimes w_2) = \sum_{i \in J} \langle w'_{3,i}, \mathcal{Y}_i(w_1, z)w_2 \rangle.$$

Therefore, for any  $v \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,

$$\sum_{i \in J} \langle w'_{3,i}, \mathcal{Y}_i(Y_{W_1}^{g_1}(v, x)w_1, z)w_2 \rangle = 0.$$

By the duality of  $\mathcal{Y}_i$ ,  $i \in J$ , on the nonempty open region given by  $0 < |z_1 - z| < |z|$ ,  $|\arg z_1 - \arg z| < \frac{\pi}{2}$ , we know that

$$\sum_{i \in I} f_{\mathcal{Y}_{i,2}}^e(z_1, z; v, w_1, w_2, w'_{3,i}) = \sum_{i \in I} \langle w'_{3,i}, \mathcal{Y}_i(Y_{W_1}^{g_1}(v, z_1)w_1, z)w_2 \rangle = 0, \quad (6.3)$$

which implies as a function on  $z_1$ ,  $\sum_{i \in J} f_{\mathcal{Y}_{i,2}}^e(z_1, z; v, w_1, w_2, w'_{3,i})$  is the identically zero function. On the nonempty open region given by  $|z_1| > |z|$ ,  $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$ , we have

$$\sum_{i \in J} \langle w'_{3,i}, Y_{W_3}^{g_3}(v, z_1)\mathcal{Y}_i(w_1, z)w_2 \rangle = \sum_{i \in J} f_{\mathcal{Y}_{i,2}}^e(z_1, z; v, w_1, w_2, w'_{3,i}) = 0, \quad (6.4)$$

which implies  $0 = \sum_{i \in J} \langle w'_{3,i}, Y_{W_3}^{g_3}(v, x)\mathcal{Y}_i(w_1, z)w_2 \rangle = \left( Y_{W_1 \boxtimes P(z)}^{(g_1 g_2)^{-1}} W_2 \right)^o (v, x)(\lambda)(w_1 \otimes w_2)$ . ■

**Remark 6.0.5.** Suppose that the direct sum of two objects in  $\mathcal{C}$  is also in  $\mathcal{C}$ . For any linear combination  $\sum_{i=1}^n \lambda_{I_i, w'_{3,i}}$ , where  $W_{3,i}$  is a  $g_1 g_2$ -twisted module in  $\mathcal{C}$ ,  $w'_{3,i} \in W'_{3,i}$ , and  $I_i$  is a twisted intertwining map of type  $\binom{W_{3,i}}{W_1 W_2}$ ,  $i = 1, \dots, n$ , we know that  $\sum_{i=1}^n \lambda_{I_i, w'_{3,i}} = \lambda_{I, w'_3}$ , where  $W_3 = \bigoplus_{i=1}^n W_{3,i} \in \text{ob } \mathcal{C}_{g_1 g_2}$ ,  $w'_3 = w'_{3,1} + \dots + w'_{3,n} \in W'_3$ , and  $I = (I_1, \dots, I_n)$  is a  $P(z)$ -twisted intertwining map of type  $\binom{W_3}{W_1 W_2}$ .

In other words, for any  $\lambda \in W_1 \boxtimes P(z) W_2$ , we can write  $\lambda = \lambda_{I, w'_3}$  for some  $W_3 \in \text{ob } \mathcal{C}$ ,  $w'_3 \in W'_3$ , and twisted intertwining map  $I$  of type  $\binom{W_3}{W_1 W_2}$ .

**Proposition 6.0.6.** The pair  $(W_1 \boxtimes P(z) W_2, Y_{W_1 \boxtimes P(z) W_2}^{(g_1 g_2)^{-1}})$  is a  $(g_1 g_2)^{-1}$ -twisted  $V$ -module.

*Proof.* For fixed  $W_3$  and  $I$ , the space spanned by all  $\lambda_{I, w'_3}$  for  $w'_3 \in W'_3$  is the image of  $W'_3$  under the linear map from  $W'_3$  to  $W_1 \boxtimes P(z) W_2$  given by  $w'_3 \mapsto \lambda_{I, w'_3}$ . This linear

map preserves the gradings, commutes with twisted vertex operators. So the space spanned by all  $\lambda_{I,w'_3}$  for  $w'_3 \in W'_3$  is a generalized  $(g_1g_2)^{-1}$ -twisted  $V$ -module. Thus,  $W_1 \boxtimes_{P(z)} W_2$ , as a sum of generalized  $(g_1g_2)^{-1}$ -twisted  $V$ -modules, is also a generalized  $(g_1g_2)^{-1}$ -twisted  $V$ -module.  $\blacksquare$

**Assumption 6.0.7.** *Throughout the remainder of this work, we assume that the following conditions for  $\mathcal{C}$  hold:*

1. *Any object in  $\mathcal{C}$  satisfies Assumption 4.0.1*
2. *For objects  $W_1$  and  $W_2$  in  $\mathcal{C}$ ,  $W_1 \boxtimes_{P(z)} W_2$  is also in  $\mathcal{C}$ .*
3. *The contragredient of an object in  $\mathcal{C}$  is also in  $\mathcal{C}$ . The direct sum of two objects in  $\mathcal{C}$  is also in  $\mathcal{C}$ .*
4. *the double contragredient of an object in  $\mathcal{C}$  is equivalent to the object. In other words,  $\dim W_{[n]} < \infty$  for any  $n \in \mathbb{C}$  and object  $W$  in  $\mathcal{C}$ .*

From the Conditions 1 and 2 in Assumption 6.0.7, we see that  $(W_1 \boxtimes_{P(z)} W_2)'$  is in  $\mathcal{C}$ . We take  $W_1 \boxtimes_{P(z)} W_2$  to be  $(W_1 \boxtimes_{P(z)} W_2)'$ . We still need to give a twisted  $P(z)$ -intertwining map  $\boxtimes_{P(z)}$  of type  $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$  or equivalently, an intertwining operator of the same type.

Let  $W$  be a  $(g_1g_2)^{-1}$ -twisted  $V$ -module in  $\mathcal{C}$  and  $f : W \rightarrow W_1 \boxtimes_{P(z)} W_2$  a  $V$ -module map. Since the double contragredient of an object in  $\mathcal{C}$  is equivalent to the object itself by Condition 3 in Assumption 6.0.7, every element of  $(W)'$  can be viewed as an element of  $W$ . For  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $w \in W$ , we define

$$\langle w, \mathcal{Y}_f(w_1, z)w_2 \rangle = (f(w))(w_1 \otimes w_2). \quad (6.5)$$

Then we define

$$\mathcal{Y}_f(w_1, x)w_2 = x^{L_{W'}(0)} e^{-(\log z)L_{W_3}(0)} \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z)L_{W_2}(0)} w_2 \quad (6.6)$$

for  $w_1 \in W_1$ ,  $w_2 \in W_2$ . We now have a linear map

$$\mathcal{Y}_f : W_1 \otimes W_2 \rightarrow W'\{x\}[\log x].$$

**Proposition 6.0.8.** *The linear map  $\mathcal{Y}_f : W_1 \otimes W_2 \rightarrow W'\{x\}[\log x]$  given by (6.5) and (6.6) above is a  $g_1 g_2$ -twisted intertwining operator of type  $\binom{W'}{W_1 W_2}$ . In particular, in the case that  $W = W_1 \boxtimes_{P(z)} W_2$  and  $f = 1_{W_1 \boxtimes_{P(z)} W_2} : W \rightarrow W_1 \boxtimes_{P(z)} W_2$  is the identity map, we obtain a  $g_1 g_2$ -twisted intertwining operator  $\mathcal{Y}_{1_{W_1 \boxtimes_{P(z)} W_2}}$  of type  $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$ .*

*Proof.* We first verify the  $L(-1)$ -derivative property.

$$\begin{aligned} & \frac{d}{dx} \mathcal{Y}_f(w_1, x)w_2 \\ &= \frac{d}{dx} x^{L_{W'}(0)} e^{-(\log z)L_{W'}(0)} \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z)L_{W_2}(0)} w_2 \\ &= x^{L_{W'}(0)-1} e^{-(\log z)L_{W'}(0)} L_{W'}(0) \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z)L_{W_2}(0)} w_2 \\ &\quad - x^{L_{W'}(0)} e^{-(\log z)L_{W'}(0)} \mathcal{Y}_f(L_{W_1}(0) x^{-L_{W_1}(0)-1} e^{(\log z)L_{W_1}(0)} w_1, z) x^{-L_{W_2}(0)} e^{(\log z)L_{W_2}(0)} w_2 \\ &\quad - x^{L_{W'}(0)} e^{-(\log z)L_{W'}(0)} \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) L_{W_2}(0) x^{-L_{W_2}(0)-1} e^{(\log z)L_{W_2}(0)} w_2 \\ &= x^{-1} x^{L_{W'}(0)} e^{-(\log z)L_{W'}(0)} \\ &\quad \cdot (L_{W'}(0) \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) - \mathcal{Y}_f(L_{W_1}(0) x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) \\ &\quad \quad - \mathcal{Y}_f(x^{-L_{W_1}(0)} e^{(\log z)L_{W_1}(0)} w_1, z) L_{W_2}(0)) \\ &\quad \cdot x^{-L_{W_2}(0)} e^{(\log z)L_{W_2}(0)} w_2. \end{aligned} \quad (6.7)$$

Since  $f$  is a  $V$ -module map,  $f(W)$  is a submodule of  $W_1 \boxtimes_{P(z)} W_2$ . By the definition of  $W_1 \boxtimes_{P(z)} W_2$ , it is spanned by elements of the form  $\lambda_{I, w'_3}$  for a  $g_1 g_2$ -twisted  $V$ -module

$W_3$ , a  $P(z)$ -intertwining map  $I$  of type  $(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix})$  and  $w'_3 \in W'_3$ . In particular, for  $w \in W$ ,

$$f(w) = \sum_{i=3}^n \lambda_{I^i, w'_i},$$

where for  $i = 3, \dots, n$ ,  $w'_i$  is an element of the contragredient module  $W'_i$  of a  $g_1 g_2$ -twisted object  $W_i$  in  $\mathcal{C}$ , and  $I^i$  a  $P(z)$ -intertwining map of type  $(\begin{smallmatrix} W_i \\ W_1 W_2 \end{smallmatrix})$ .

Let  $\mathcal{Y}^i$  be the intertwining operator of type  $(\begin{smallmatrix} W_i \\ W_1 W_2 \end{smallmatrix})$  such that  $I^i = \mathcal{Y}^i(\cdot, z)$ . Then we have

$$\begin{aligned} \langle w, \mathcal{Y}_f(w_1, z)w_2 \rangle &= (f(w))(w_1 \otimes w_2) \\ &= \sum_{i=3}^n \lambda_{I^i, w'_i}(w_1 \otimes w_2) \\ &= \sum_{i=3}^n \langle w'_i, I^i(w_1 \otimes w_2) \rangle \\ &= \sum_{i=3}^n \langle w'_i, \mathcal{Y}^i(w_1, z)w_2 \rangle \end{aligned}$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . Also,

$$\begin{aligned} (f(L_W(0)w))(w_1 \otimes w_2) &= ((L_{W_1 \boxtimes_{P(z)} W_2}(0)f(w))(w_1 \otimes w_2) \\ &= \sum_{i=3}^n (L_{W_1 \boxtimes_{P(z)} W_2}(0)\lambda_{I^i, w'_i})(w_1 \otimes w_2) \\ &= \sum_{i=3}^n \text{Res}_x x (Y_{W'}^{(g_1 g_2)^{-1}}(\omega, x)\lambda_{I^i, w'_i})(w_1 \otimes w_2) \\ &= \sum_{i=3}^n \text{Res}_x x (\lambda_{I^i, Y_{W'_i}^{(g_1 g_2)^{-1}}(\omega, x)w'_i})(w_1 \otimes w_2) \\ &= \sum_{i=3}^n \lambda_{I^i, L_{W'_i}(0)w'_i}(w_1 \otimes w_2). \end{aligned}$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . So we have

$$f(L_W(0)w) = \sum_{i=3}^n \lambda_{I^i, L_{W'_i}(0)w'_i}.$$

Thus for  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,

$$\begin{aligned} & \langle w, (L_{W'}(0)\mathcal{Y}_f(w_1, z) - \mathcal{Y}_f(L_{W_1}(0)w_1, z) - \mathcal{Y}_f(w_1, z)L_{W_2}(0))w_2 \rangle \\ &= \langle L_W(0)w, \mathcal{Y}_f(w_1, z)w_2 \rangle - \langle w, \mathcal{Y}_f(L_{W_1}(0)w_1, z)w_2 \rangle - \langle w, \mathcal{Y}_f(w_1, z)L_{W_2}(0)w_2 \rangle \\ &= (f(L_W(0)w))(w_1 \otimes w_2) - (f(w))(L_{W_1}(0)w_1 \otimes w_2) - (f(w))(w_1 \otimes L_{W_2}(0)w_2) \\ &= \sum_{i=3}^n \lambda_{I^i, L_{W'_i}(0)w'_i}(w_1 \otimes w_2) - \sum_{i=3}^n \lambda_{I, w'_i}(L_{W_1}(0)w_1 \otimes w_2) - \sum_{i=3}^n \lambda_{I, w'_i}(w_1 \otimes L_{W_2}(0)w_2) \\ &= \sum_{i=3}^n \langle L_{W'_i}(0)w'_i, \mathcal{Y}^i(w_1, z)w_2 \rangle - \sum_{i=3}^n \langle w'_i, \mathcal{Y}^i(L_{W_1}(0)w_1, z)w_2 \rangle \\ & \quad - \sum_{i=3}^n \langle w'_i, \mathcal{Y}^i(w_1, z)L_{W_2}(0)w_2 \rangle \\ &= \sum_{i=3}^n \langle w'_i, (L_{W_i}(0)\mathcal{Y}^i(w_1, z) - \mathcal{Y}^i(L_{W_1}(0)w_1, z) - \mathcal{Y}^i(w_1, z)L_{W_2}(0))w_2 \rangle \\ &= \sum_{i=3}^n z \langle w'_i, \mathcal{Y}^i(L_{W_1}(-1)w_1, z)w_2 \rangle \\ &= z \sum_{i=3}^n \lambda_{I^i, w'_i}((L_{W_1}(-1)w_1 \otimes w_2) \\ &= z(f(w))(L_{W_1}(-1)w_1 \otimes w_2) \\ &= z \langle w, \mathcal{Y}_f(L_{W_1}(-1)w_1, z)w_2 \rangle, \end{aligned}$$

where we have used the  $L(0)$ -commutator formula for the twisted intertwining operators  $\mathcal{Y}^i$ . Since  $w \in W$  and  $w_2 \in W_2$  are arbitrary, we obtain

$$L_W(0)\mathcal{Y}_f(w_1, z) - \mathcal{Y}_f(L_{W_1}(0)w_1, z) - \mathcal{Y}_f(w_1, z)L_{W_2}(0) = z\mathcal{Y}_f(L_{W_1}(-1)w_1, z) \quad (6.8)$$

for  $w_1 \in W_1$ .

Using (6.8), we see that the right-hand side of (6.7) is equal to

$$\begin{aligned}
& x^{-1}x^{L_W(0)}e^{-(\log z)L_W(0)}z\mathcal{Y}_f(L_{W_1}(-1)x^{-L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)x^{-L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \\
&= x^{L_W(0)}e^{-(\log z)L_W(0)}\mathcal{Y}_f(x^{-L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}L_{W_1}(-1)w_1, z)x^{-L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \\
&= \mathcal{Y}_f(L_{W_1}(-1)w_1, x)w_2,
\end{aligned}$$

proving the  $L(-1)$ -derivative property.

For  $v \in V$ ,  $w_1 \in W_1$  and  $w_2 \in W_2$ , we have

$$\begin{aligned}
(f(Y_W^{(g_1g_2)^{-1}}(v, x)w))(w_1 \otimes w_2) &= (Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1g_2)^{-1}}(v, x)f(w))(w_1 \otimes w_2) \\
&= \sum_{i=3}^n (Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1g_2)^{-1}}(v, x)\lambda_{I^i, w'_i})(w_1 \otimes w_2) \\
&= \sum_{i=3}^n \lambda_{I^i, Y_{W_i}^{(g_1g_2)^{-1}}(v, x)w'_i}(w_1 \otimes w_2).
\end{aligned}$$

Then we obtain

$$f(Y_W^{(g_1g_2)^{-1}}(v, x)w) = \sum_{i=3}^n \lambda_{I^i, Y_{W_i}^{(g_1g_2)^{-1}}(v, x)w'_i}$$

For a  $g_1g_2$ -twisted  $V$ -module  $W_3$  in  $\mathcal{C}$ , a  $P(z)$ -intertwining map  $I$  of type  $\binom{W_3}{W_1W_2}$ ,  $u \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $w'_3 \in W'_3$ , we have

$$\begin{aligned}
& \langle w, Y_{W'_3}^{g_1g_2}(u, z_1)\mathcal{Y}_f(w_1, z_2)w_2 \rangle \\
&= \langle w, Y_{W'_3}^{g_1g_2}(u, z_1)e^{(\log z_2)L_{W'_3}(0)}e^{-(\log z)L_{W'_3}(0)} \\
&\quad \cdot \mathcal{Y}_f(e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \rangle \\
&= \langle e^{(\log z_2)L_W(0)}e^{-(\log z)L_W(0)}Y_W^{(g_1g_2)^{-1}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w, \\
&\quad \mathcal{Y}_f(e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1, z)e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2 \rangle \\
&= (f(e^{(\log z_2)L_W(0)}e^{-(\log z)L_W(0)}Y_W^{(g_1g_2)^{-1}}(e^{z_1L_V(1)}(-z_1^{-2})^{L_V(0)}u, z_1^{-1})w) \\
&\quad (e^{-(\log z_2)L_{W_1}(0)}e^{(\log z)L_{W_1}(0)}w_1 \otimes e^{-(\log z_2)L_{W_2}(0)}e^{(\log z)L_{W_2}(0)}w_2)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=3}^n \lambda_{I^i, e^{(\log z_2)L_{W'_i(0)}} e^{-(\log z)L_{W'_i(0)}} Y_{W'_i}^{(g_1 g_2)^{-1}}(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)} u, z_1^{-1}) w'_i} \\
&\quad (e^{-(\log z_2)L_{W_1(0)}} e^{(\log z)L_{W_1(0)}} w_1 \otimes e^{-(\log z_2)L_{W_2(0)}} e^{(\log z)L_{W_2(0)}} w_2) \\
&= \sum_{i=3}^n \langle e^{(\log z_2)L_{W'_i(0)}} e^{-(\log z)L_{W'_i(0)}} Y_{W'_i}^{(g_1 g_2)^{-1}}(e^{z_1 L_V(1)}(-z_1^{-2})^{L_V(0)} u, z_1^{-1}) w'_i, \\
&\quad \mathcal{Y}^i(e^{-(\log z_2)L_{W_1(0)}} e^{(\log z)L_{W_1(0)}} w_1, z) e^{-(\log z_2)L_{W_2(0)}} e^{(\log z)L_{W_2(0)}} w_2 \rangle \\
&= \sum_{i=3}^n \langle w'_i, Y_{W'_i}^{g_1 g_2}(u, z_1) e^{(\log z_2)L_{W'_i(0)}} e^{-(\log z)L_{W'_i(0)}} \cdot \\
&\quad \cdot \mathcal{Y}^i(e^{-(\log z_2)L_{W_1(0)}} e^{(\log z)L_{W_1(0)}} w_1, z) e^{-(\log z_2)L_{W_2(0)}} e^{(\log z)L_{W_2(0)}} w_2 \rangle \\
&= \sum_{i=3}^n \langle w'_i, Y_{W'_i}^{g_1 g_2}(u, z_1) \mathcal{Y}^i(w_1, z_2) w_2 \rangle. \tag{6.9}
\end{aligned}$$

Similarly, we have

$$\langle w, \mathcal{Y}_f(w_1, z_2) Y_{W_2}^{g_2}(u, z_1) w_2 \rangle = \sum_{i=3}^n \langle w'_i, \mathcal{Y}^i(w_1, z_2) Y_{W_2}^{g_2}(u, z_1) w_2 \rangle \tag{6.10}$$

and

$$\langle w, \mathcal{Y}_f(Y_{W_2}^{g_1}(u, z_1 - z_2) w_1, z_2) w_2 \rangle = \sum_{i=3}^n \langle w'_i, \mathcal{Y}^i(Y_{W_1}^{g_1}(u, z_1 - z_2) w_1, z_2) w_2 \rangle. \tag{6.11}$$

Since  $\mathcal{Y}^i$  for  $i = 1, \dots, n$  are twisted intertwining operators, the duality property for  $\mathcal{Y}_f$  follows from (6.9), (6.10), (6.11) and the duality properties for  $\mathcal{Y}^i$ .

The convergence for products of more than two operators follows from the formula

$$\begin{aligned}
&\langle w, Y_{W'_1}^{g_1 g_2}(u_1, z_1) \cdots Y_{W'_k}^{g_1 g_2}(u_{k-1}, z_{k-1}) \mathcal{Y}_f(w_1, z_k) w_2 \rangle \\
&= \sum_{i=3}^n \langle w'_i, Y_{W'_i}^{g_1 g_2}(u_1, z_1) \cdots Y_{W'_i}^{g_1 g_2}(u_{k-1}, z_{k-1}) \mathcal{Y}^i(w_1, z_k) w_2 \rangle, \tag{6.12}
\end{aligned}$$

whose proof is the same as that of (6.9). Since  $\mathcal{Y}^i$  are twisted intertwining operators, the fact that  $f_{\mathcal{Y}^i, k}$  satisfies Condition 3.0.13 for  $(k-1, 1, \{0\})$  implies the fact that

the function that (6.12) converges to satisfies Condition 3.0.13 for  $(k-1, 1, \{0\})$ .

□

Let  $\boxtimes_{P(z)} = \mathcal{Y}_{1_{W_1 \boxtimes_{P(z)} W_2}}(\cdot, z)$ . Then  $\boxtimes_{P(z)}$  is a  $P(z)$ -intertwining map of type  $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$ . Let

$$w_1 \boxtimes_{P(z)} w_2 = \boxtimes_{P(z)}(w_1 \otimes w_2) = \mathcal{Y}_{1_{W_1 \boxtimes_{P(z)} W_2}}(w_1, z)w_2 \in \overline{W_1 \boxtimes_{P(z)} W_2}$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . We call  $w_1 \boxtimes_{P(z)} w_2$  the  $P(z)$ -tensor product of  $w_1$  and  $w_2$ .

By (6.5), we have

$$\lambda(w_1 \otimes w_2) = \langle \lambda, w_1 \boxtimes_{P(z)} w_2 \rangle \quad (6.13)$$

for  $\lambda \in W_1 \boxtimes_{P(z)} W_2$ ,  $w_1 \in W_1$  and  $w_2 \in W_2$ .

**Theorem 6.0.9.** *The pair  $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$  is a  $P(z)$ -tensor product of  $W_1$  and  $W_2$  in  $\mathcal{C}$ .*

*Proof.* First, by Assumption 6.0.7, we know that  $W_1 \boxtimes_{P(z)} W_2$  is indeed an object in  $\mathcal{C}$ .

Let  $(W_3, I)$  be a  $P(z)$ -product of  $W_1$  and  $W_2$  with  $W_3 \in \text{ob } \mathcal{C}$ . Then we have a module map  $g : W_3' \rightarrow W_1 \boxtimes_{P(z)} W_2$  given by  $g(w_3') = \lambda_{I, w_3'}$  for  $w_3' \in W_3'$ . By definition, we have  $(g(w_3'))(w_1 \otimes w_2) = \lambda_{I, w_3'}(w_1 \otimes w_2) = \langle w_3', I(w_1 \otimes w_2) \rangle$  for  $w_1 \in W_1$  and  $w_2 \in W_2$ . The adjoint of this module map is a module map  $f : W_1 \boxtimes_{P(z)} W_2 \rightarrow W_3$ . By definitions and (6.13),

$$\begin{aligned} \langle w_3', (\bar{f} \circ \boxtimes_{P(z)})(w_1 \otimes w_2) \rangle &= \langle w_3', \bar{f}(w_1 \boxtimes_{P(z)} w_2) \rangle \\ &= \langle g(w_3'), w_1 \boxtimes_{P(z)} w_2 \rangle \\ &= (g(w_3'))(w_1 \otimes w_2) \\ &= \langle w_3', I(w_1 \otimes w_2) \rangle. \end{aligned}$$

So we obtain  $\bar{f} \circ \boxtimes_{P(z)} = I$ . ■

We have assigned each object  $(W_1, W_2)$  in the category  $\mathcal{C} \times \mathcal{C}$  an object  $W_1 \boxtimes_{P(z)} W_2$  in  $\mathcal{C}$ . To obtain a functor from  $\mathcal{C} \times \mathcal{C}$  to  $\mathcal{C}$ , we still need to assign a morphism  $(f_1, f_2)$  in  $\mathcal{C} \times \mathcal{C}$  a morphism  $f_1 \boxtimes_{P(z)} f_2$  in  $\mathcal{C}$ .

Let  $W_1, \widetilde{W}_1$  be  $g_1$ -twisted  $V$ -modules in  $\mathcal{C}$  and  $W_2, \widetilde{W}_2$   $g_1$ -twisted  $V$ -modules in  $\mathcal{C}$ . Let  $f_1 : W_1 \rightarrow \widetilde{W}_1$  and  $f_2 : W_2 \rightarrow \widetilde{W}_2$  be module maps. Let  $\widetilde{\mathcal{Y}}$  be the intertwining operator of type  $\left(\begin{smallmatrix} \widetilde{W}_1 \boxtimes_{P(z)} \widetilde{W}_2 \\ \widetilde{W}_1 \widetilde{W}_2 \end{smallmatrix}\right)$  such that  $\tilde{w}_1 \boxtimes_{P(z)} \tilde{w}_2 = \widetilde{\mathcal{Y}}(\tilde{w}_1, z)\tilde{w}_2$ . Since  $f_1$  and  $f_2$  are module maps,  $\mathcal{Y} = \widetilde{\mathcal{Y}} \circ (f_1 \otimes f_2)$  is an intertwining operator of type  $\left(\begin{smallmatrix} \widetilde{W}_1 \boxtimes_{P(z)} \widetilde{W}_2 \\ W_1 W_2 \end{smallmatrix}\right)$ . Then  $I = \mathcal{Y}(\cdot, z)$  is a  $P(z)$ -intertwining operator of the same type. Hence, we have a  $P(z)$ -product  $(\widetilde{W}_1 \boxtimes_{P(z)} \widetilde{W}_2, I)$  of  $W_1$  and  $W_2$ . By the universal property of the tensor product  $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ , there exist a unique module map  $f : W_1 \boxtimes_{P(z)} W_2 \rightarrow \widetilde{W}_1 \boxtimes_{P(z)} \widetilde{W}_2$  such that  $I = \bar{f} \circ \boxtimes_{P(z)}$ . We define this module map  $f$  to be the  $P(z)$ -tensor product of  $f_1$  and  $f_2$  and denote it by  $f_1 \boxtimes_{P(z)} f_2$ .

**Theorem 6.0.10.** *The assignments given by  $(W_1, W_2) \mapsto W_1 \boxtimes_{P(z)} W_2$  and  $(f_1, f_2) \mapsto f_1 \boxtimes_{P(z)} f_2$  above is a functor from  $\mathcal{C} \times \mathcal{C}$  to  $\mathcal{C}$ .*

*Proof.* It is easy to verify  $1_{w_1} \boxtimes_{P(z)} 1_{w_2} = 1_{W_1 \boxtimes_{P(z)} W_2}$  and  $(f_1 \boxtimes_{P(z)} f_2) \circ (g_1 \boxtimes_{P(z)} g_2) = (f_1 g_1) \boxtimes_{P(z)} (f_2 g_2)$  by using the construction of the tensor products of module maps. We omit the details of the proofs. ■

We call this functor the  $P(z)$ -tensor product bifunctor for  $\mathcal{C}$ .

We now give a result on Condition 2 in Assumption 6.0.7.

**Theorem 6.0.11.** *Let  $G \leq \text{Aut}(V)$ . Assume that the following conditions are satisfied:*

1. *For  $g \in G$ , there are only finitely many irreducible grading-restricted  $g$ -twisted  $V$ -modules.*
2. *For  $g \in G$ , every grading-restricted  $g$ -twisted  $V$ -module is completely reducible.*

3. For  $g_1, g_2 \in G$  and grading-restricted  $g_1$ -,  $g_2$ -,  $g_1g_2$ -twisted  $V$ -modules  $W_1, W_2, W_3$ , the fusion rule  $N_{W_1W_2}^{W_3} := \dim \mathcal{V}_{W_1W_2}^{W_3}$  is finite.

Then  $W_1 \boxtimes_{P(z)} W_2$  is in  $\mathcal{GM}_{gr}(G)$  for any objects  $W_1$  and  $W_2$  in  $\mathcal{GM}_{gr}(G)$ .

*Proof.* Let  $W_1$  and  $W_2$  be  $g_1$ - and  $g_2$ -twisted  $V$ -modules in  $\mathcal{GM}_{gr}(G)$ . Then  $W_1 \boxtimes_{P(z)} W_2$  is a generalized  $(g_1g_2)^{-1}$ -twisted  $V$ -module. From the construction of  $W_1 \boxtimes_{P(z)} W_2$ , it is a sum of grading-restricted  $(g_1g_2)^{-1}$ -twisted  $V$ -module. By Condition 2,  $W_1 \boxtimes_{P(z)} W_2$  must be a direct sum of irreducible grading-restricted  $(g_1g_2)^{-1}$ -twisted  $V$ -modules. But by Condition 1, there are only finitely many irreducible grading-restricted  $(g_1g_2)^{-1}$ -twisted  $V$ -modules. If  $W_1 \boxtimes_{P(z)} W_2$  is an infinite direct sum of irreducible grading-restricted  $(g_1g_2)^{-1}$ -twisted  $V$ -modules, at least one irreducible grading-restricted  $(g_1g_2)^{-1}$ -twisted  $V$ -module  $W_3$  has infinitely many copies in this decomposition of  $W_1 \boxtimes_{P(z)} W_2$ . But then we have infinitely many linearly independent injective  $V$ -module maps from  $W_3$  to the  $W_1 \boxtimes_{P(z)} W_2$ . But by Proposition 6.0.8, these infinite injective  $V$ -module maps give linearly independent twisted intertwining operator of type  $\left( \begin{smallmatrix} W_3 \\ W_1W_2 \end{smallmatrix} \right)$ . Thus the fusion rule  $N_{W_1W_2}^{W_3}$  is  $\infty$ . By Condition 3, this is a contradiction. So  $W_1 \boxtimes_{P(z)} W_2$  must be a finite direct sum of irreducible grading-restricted  $(g_1g_2)^{-1}$ -twisted  $V$ -modules. In particular, it is grading restricted. ■

**Corollary 6.0.12.** *Assume that the three conditions in Theorem 6.0.11 are satisfied. Also assume that  $L_W(0)_N$ , the nilpotent part of  $L(0)_W$ , is nilpotent on  $W$ , for any  $W$  in  $\mathcal{GM}_{gr}(G)$ . Then, the category  $\mathcal{GM}_{gr}(G)$  satisfies Assumption 6.0.7.*

*Proof.* Theorem 6.0.11 shows that Condition 2 holds. Conditions 1, 3 and 4 are clearly holds for the category  $\mathcal{GM}_{gr}(G)$ . ■

For the same  $W_1$  and  $W_2$ , let  $\mathcal{Y}$  be the twisted intertwining operator of type  $\left( \begin{smallmatrix} \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1 \\ \phi_{g_1}(W_2)W_1 \end{smallmatrix} \right)$  such that  $w_2 \boxtimes_{P(-z)} w_1 = \mathcal{Y}(w_2, -z)w_1$  for  $w_1 \in W_1, w_2 \in \phi_{g_1}(W_2) =$

$W_2$ . Then by Theorem 4.0.4,  $\Omega_+(\mathcal{Y})$  is a twisted intertwining operator of type

$$\begin{pmatrix} \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1 \\ W_1 \phi_{g_1}^{-1}(\phi_{g_1}(W_2)) \end{pmatrix} = \begin{pmatrix} \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1 \\ W_1 W_2 \end{pmatrix}.$$

In particular, the pair  $(\phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1, \Omega_+(\mathcal{Y})(\cdot, z)\cdot)$  is a  $P(z)$ -product of  $W_1$  and  $W_2$ . By the universal property of the tensor product  $W_1 \boxtimes_{P(z)} W_2$ , there exists a unique  $g_1 g_2$ -twisted  $V$ -module map

$$\mathcal{R}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \rightarrow \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1$$

such that

$$\Omega(\mathcal{Y})(\cdot, z)\cdot = \overline{\mathcal{R}}_{P(z)} \circ \boxtimes_{P(z)},$$

where  $\overline{\mathcal{R}}_{P(z)}$  is the natural extension of  $\mathcal{R}_{P(z)}$ . The  $g_1 g_2$ -twisted  $V$ -module map  $\mathcal{R}_{P(z)}$  has an inverse

$$\mathcal{R}_{P(z)}^{-1} : \phi_{g_1}(W_2) \boxtimes_{P(-z)} W_1 \rightarrow W_1 \boxtimes_{P(z)} W_2$$

constructed in the same as above except that we use  $\Omega_-$  instead of  $\Omega_+$ . Then we obtain a natural isomorphism  $\mathcal{R}_{P(z)}$  called the  *$G$ -crossed commutativity isomorphisms*.

As in [H7] and [HLZ3], we also have parallel transport isomorphisms. Let  $z_1, z_2 \in \mathbb{C}^\times$  and  $\gamma$  a path in  $\mathbb{C}^\times$  from  $z_1$  to  $z_2$ . We denote the homotopy class of  $\gamma$  by  $[\gamma]$ . For the same  $W_1$  and  $W_2$ , let  $\mathcal{Y}$  be the twisted intertwining operator of type  $\begin{pmatrix} W_1 \boxtimes_{P(z_2)} W_2 \\ W_1 W_2 \end{pmatrix}$  such that  $w_1 \boxtimes_{P(z_2)} w_2 = \mathcal{Y}(w_1, z)w_2$  for  $w_1 \in W_1, w_2 \in W_2$ . Then  $(W_1 \boxtimes_{P(z_2)} W_2, \mathcal{Y}(\cdot, z_1)\cdot)$  is a  $P(z_1)$ -product of  $W_1$  and  $W_2$ . By the universal property of the  $P(z_1)$ -tensor product  $W_1 \boxtimes_{P(z_1)} W_2$ , there exists a unique  $g_1 g_2$ -twisted  $V$ -module map

$$\mathcal{T}_{[\gamma]} : W_1 \boxtimes_{P(z_1)} W_2 \rightarrow W_1 \boxtimes_{P(z_2)} W_2 \tag{6.14}$$

such that  $\overline{\mathcal{T}}_{[\gamma]} \circ \boxtimes_{P(z_1)} = \boxtimes_{P(z_2)}$ . The  $g_1 g_2$ -twisted twisted  $V$ -module map  $\mathcal{T}_{[\gamma]}$  is

invertible since the same construction also gives a  $g_1g_2$ -twisted  $V$ -module map

$$\mathcal{T}_{[\gamma^{-1}]} : W_1 \boxtimes_{P(z_2)} W_2 \rightarrow W_1 \boxtimes_{P(z_1)} W_2$$

which is clearly the inverse of  $\mathcal{T}_{[\gamma]}$ . Thus the natural transformation  $\mathcal{T}_{[\gamma]}$  is a natural isomorphism called the *parallel transport isomorphism from  $z_1$  to  $z_2$  along  $[\gamma]$* .

Let  $\gamma$  be the path  $[0, 1] \ni t \mapsto e^{-\pi it + \pi i}$ . For the same  $W_1$  and  $W_2$ , we define the  *$G$ -crossed braiding isomorphism  $\mathcal{R} : W_1 \boxtimes_{P(1)} W_2 \rightarrow \phi_{g_1}(W_2) \boxtimes_{P(1)} W_1$*  by

$$\mathcal{R} = \mathcal{T}_{[\gamma]} \circ \mathcal{R}_{P(1)}.$$

## Chapter 7

# Compatibility Condition and $\mathcal{C}$ -Embeddability Condition

For a fixed  $z \in \mathbb{C}_\times$ , we need to study multivalued analytic functions on the region  $F_n(\mathbb{C} - \{0, z\})$  for  $n \in \mathbb{Z}_+$ , and its subregions

$$\Omega_{m,k,l}(z) = \left\{ \begin{array}{l} (z_1, \dots, z_{m+k+l}) \\ \in \mathbb{C}^{m+k+l} \end{array} \left| \begin{array}{l} |z| < |z_m| < \dots < |z_1|, \text{ if } m > 0, \\ 0 < |z_{m+k} - z| < \dots < |z_{m+1} - z| < |z|, \text{ if } k > 0, \\ 0 < |z_{m+k+l}| < \dots < |z_{m+k+1}| < |z|, \text{ if } l > 0, \\ |z_{m+1} - z| + |z_{m+k+1}| < |z|, \text{ if } k > 0, l > 0, \\ |z_{m+1} - z| + |z| < |z_m|, \text{ if } m > 0, k > 0 \end{array} \right. \right\},$$

$$\Omega_{m,k,l}^{(1)}(z) = \left\{ \begin{array}{l} (z_1, \dots, z_{m+k+l}) \\ \in \Omega_{m,k,l}(z) \end{array} \left| \begin{array}{l} |\arg(z_j - z) - \arg(z_j)| < \frac{\pi}{2}, \\ \quad j = 1, \dots, m, \text{ if } m > 0 \\ |\arg(z_j) - \arg(z)| < \frac{\pi}{2}, \\ \quad j = m + 1, \dots, m + k, \text{ if } k > 0 \\ -\frac{3\pi}{2} < \arg(z_j - z) - \arg(z) < -\frac{\pi}{2}, \\ \quad j = m + k + 1, \dots, m + k + l, \text{ if } l > 0 \end{array} \right. \right\},$$

$$\Omega_{m,k,l}^{(2)}(z) = \left\{ \begin{array}{l} (z_1, \dots, z_{m+k+l}) \\ \in \Omega_{m,k,l}(z) \end{array} \left| \begin{array}{l} |\arg(z_j - z) - \arg(z_j)| < \frac{\pi}{2}, \\ j = 1, \dots, m, \text{ if } m > 0 \\ |\arg(z_j) - \arg(z)| < \frac{\pi}{2}, \\ j = m + 1, \dots, m + k, \text{ if } k > 0 \\ \frac{\pi}{2} < \arg(z_j - z) - \arg(z) < \frac{3\pi}{2}, \\ j = m+k+1, \dots, m+k+l, \text{ if } l > 0 \end{array} \right. \right\}.$$

for  $m, k, l \in \mathbb{N}$  with  $m + k + l > 0$ .

**Proposition 7.0.1.** *Let  $W_1, W_2, W_3$  be  $g_1^-$ ,  $g_2^-$ ,  $g_1g_2$ -twisted  $V$ -modules, respectively. Let  $w'_3 \in W'_3$ . Let  $z \in \mathbb{C}^\times$  and  $I$  be a twisted  $P(z)$ -intertwining map of type  $\left(\begin{smallmatrix} W_3 \\ W_1W_2 \end{smallmatrix}\right)$ . Then, the element  $\lambda_{I,w'_3}$  has the following property: For  $n \in \mathbb{N}$ ,  $u_1, \dots, u_n \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , there exists a multivalued analytic function*

$$f_n(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda_{I,w'_3}) \quad (7.1)$$

on  $F_n(\mathbb{C} - \{0, z\})$  satisfying Condition 3.0.13 for  $(n, 0, \{0, z\})$ , with a preferred single valued branch

$$f_n^e(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda_{I,w'_3}) \quad (7.2)$$

on  $F_{n,0}^0(\mathbb{C} - \{0, z\})$ , satisfying the following:

1. For any  $u_1 \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , the series

$$\begin{aligned} & \lambda_{I,w'_3} \left( Y^{g_1}(u_1, z_1 - z) w_1 \otimes w_2 \right) \\ &= \lambda_{I,w'_3} \left( Y^{g_1}(u_1, x) w_1 \otimes w_2 \right) \Big|_{x^n = e^{n l_0(z_1 - z)}, \log x = l_0(z_1 - z)} \end{aligned} \quad (7.3)$$

is absolutely convergent on the region  $\Omega_{0,1,0}(z)$ . Moreover, it is convergent to  $f_1^e(z_1; u_1$ ,

$w_1, w_2; \lambda_{I, w'_3}$ ) on the region  $\Omega_{0,1,0}^{(1)}(z) = \Omega_{0,1,0}^{(2)}(z)$ .

2. For any  $l \in \mathbb{N}$ ,  $u_1, \dots, u_l \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , the multiple series

$$\begin{aligned} & \lambda_{I, w'_3} \left( w_1 \otimes Y^{g_2}(u_1, z_1) \cdots Y^{g_2}(u_l, z_l) w_2 \right) \\ &= \lambda_{I, w'_3} \left( w_1 \otimes Y^{g_2}(u_1, x_1) \cdots Y^{g_2}(u_l, x_l) w_2 \right) \Big|_{x_i^n = e^{nl_0(z_1)}, \log(x_i) = l_0(z_1), i=1, \dots, l} \end{aligned} \quad (7.4)$$

in powers and logarithms of  $z_1, \dots, z_l$  is absolutely convergent on the region  $\Omega_{0,0,l}(z)$ . Moreover, it is absolutely convergent to (7.2) on the region  $\Omega_{0,0,l}^{(1)}(z)$  and to

$$f_l^{b_{1,z}^{-1} \cdots b_{l,z}^{-1}}(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda)$$

on the region  $\Omega_{0,0,l}^{(2)}(z)$  (See Remark 7.0.4 for the definition of  $b_{1,z}, \dots, b_{l,z}$ ).

3. The multivalued analytic function in  $z_1, \dots, z_{n+1}$ ,

$$f_n \left( \frac{z_1 z}{z_{n+1}}, \dots, \frac{z_n z}{z_{n+1}}; u_1, \dots, u_n, w_1, w_2; \lambda_{I, w'_3} \right), \quad (7.5)$$

satisfies Condition 3.0.13 for  $(n, 1, \{0\})$ .

*Proof.* This result can be easily verified by using the definitions of  $\lambda_{I, w'_3}$  and  $P(z)$ -intertwining maps and the properties of twisted intertwining operators including Lemmas 3.0.33 and 3.0.37. We omit the details.  $\blacksquare$

Let  $g_1$  and  $g_2$  be automorphisms of  $V$ , and  $W_1, W_2$  be  $g_1$ -,  $g_2$ -twisted  $V$ -modules, respectively. Let  $z \in \mathbb{C}^\times$ . Motivated by Proposition 7.0.1, we formulate the following condition for  $\lambda \in (W_1 \otimes W_2)^*$ :

**$P(z)$ -Compatibility condition** An element  $\lambda \in (W_1 \otimes W_2)^*$  is said to be  $P(z)$ -compatible with respect to  $W_1$  and  $W_2$ , if for  $n \in \mathbb{N}$ ,  $u_1, \dots, u_n \in V$ ,  $w_1 \in W_1$ ,

and  $w_2 \in W_2$ , there exists a multivalued analytic function

$$f_n(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda) \quad (7.6)$$

on  $F_n(\mathbb{C} - \{0, z\})$  satisfying Condition 3.0.13 for  $(n, 0, \{0, z\})$ , with a preferred maximally extended single-valued branch  $f_n^e(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda)$  on  $F_{n,0}^0(\mathbb{C} - \{0, z\})$ , satisfying the following:

1. For any  $u_1 \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , the series

$$\begin{aligned} & \lambda\left(Y^{g_1}(u_1, z_1 - z)w_1 \otimes w_2\right) \\ &= \lambda\left(Y^{g_1}(u_1, x)w_1 \otimes w_2\right) \Big|_{x^n=e^{nl_0(z_1-z)}, \log x=l_0(z_1-z)} \end{aligned} \quad (7.7)$$

is absolutely convergent on the region  $\Omega_{0,1,0}(z)$ . Moreover, it is convergent to  $f_1^e(z_1; u_1, w_1, w_2; \lambda)$  on the region  $\Omega_{0,1,0}^{(1)}(z) = \Omega_{0,1,0}^{(2)}(z)$ .

2. For any  $l \in \mathbb{N}$ ,  $u_1, \dots, u_l \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , the multiple series

$$\begin{aligned} & \lambda\left(w_1 \otimes Y^{g_2}(u_1, z_1) \cdots Y^{g_2}(u_l, z_l)w_2\right) \\ &= \lambda\left(w_1 \otimes Y^{g_2}(u_1, x_1) \cdots Y^{g_2}(u_l, x_l)w_2\right) \Big|_{x_i^n=e^{nl_0(z_1)}, \log(x_i)=l_0(z_1), i=1,\dots,l} \end{aligned} \quad (7.8)$$

in powers and logarithms of  $z_1, \dots, z_l$  is absolutely convergent on the region  $\Omega_{0,0,l}(z)$ . Moreover, it is absolutely convergent to

$$f_l^e(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda) \quad (7.9)$$

on the region  $\Omega_{0,0,l}^{(1)}(z)$  and to

$$f_l^{b_{1,z}^{-1} \cdots b_{l,z}^{-1}}(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda) \quad (7.10)$$

on the region  $\Omega_{0,0,l}^{(2)}(z)$ . (See Remark 7.0.4 for the definition of  $b_{1,z}, \dots, b_{l,z}$ .)

3. As a function on  $n + 1$  variables  $z_1, \dots, z_{n+1}$ , the function

$$f_n \left( \frac{z_1 z}{z_{n+1}}, \dots, \frac{z_n z}{z_{n+1}}; u_1, \dots, u_n, w_1, w_2; \lambda \right)$$

satisfies parts (d) and (e) of Condition 3.0.13 for  $(n, 1, \{0\})$ .

We denote the subspace of  $(W_1 \otimes W_2)^*$  consisting of functionals which are  $P(z)$ -compatible with respect to  $W_1$  and  $W_2$  as  $\text{COMP}_{P(z)}(W_1, W_2)$ , or COMP for short.

**Remark 7.0.2.** *If  $\lambda$  is  $P(z)$ -compatible, then  $f_n \left( \frac{z_1 z}{z_{n+1}}, \dots, \frac{z_n z}{z_{n+1}}; u_1, \dots, u_n, w_1, w_2; \lambda \right)$  satisfies the whole Condition 3.0.13 for  $(n, 1, \{0\})$ , because of Proposition 3.0.21.*

**Remark 7.0.3.** *In part 2. of Proposition 7.0.1 and part 2. of the  $P(z)$ -compatibility condition, the reason that we involve two different regions  $\Omega_{0,0,l}^{(1)}(z)$  and  $\Omega_{0,0,l}^{(2)}(z)$  is because either of these two regions could be empty. Notice that here  $z$  is a **fixed** nonzero complex number. Actually, when  $\arg(z) \in [0, \pi/2]$ , the region  $\Omega_{0,0,l}^{(1)}(z)$  is empty. When  $\arg(z) \in [3\pi/2, 2\pi)$ , the region  $\Omega_{0,0,l}^{(2)}(z)$  is empty. When  $\Omega_{0,0,l}^{(1)}(z)$  and  $\Omega_{0,0,l}^{(2)}(z)$  are both nonempty, the absolute convergence of (7.8) on these two regions are equivalent.*

**Remark 7.0.4.** *Because of the domain of  $f_n^e(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda)$ , i.e.  $F_{n,0}^0(\mathbb{C} - \{0, z\})$ , and the fact that  $f_n$  satisfies Condition 3.0.13 for  $(n, 0, \{0, z\})$ , branches of  $f_n$  can be indexed and acted by elements in the fundamental group of the space*

$$F_{n,0}(\mathbb{C} - \{0, z\}) = \prod_{i=1}^n \{z_i \in \mathbb{C} \mid z_i \neq 0, z_i \neq z\}.$$

*A set of generators of this fundamental group can be chosen as  $b_{i,0}, b_{i,z}, i = 1, \dots, n$ . For each  $i$ , the elements  $b_{i,0}$  and  $b_{i,z}$  corresponds to  $b_{13}$  and  $b_{12}$  in the definition of*

twisted intertwining operator, and they freely generate  $\pi_1(\{z_i \in \mathbb{C} \mid z_i \neq 0, z_i \neq z\})$ .

Notice that

$$\pi_1 \left( \prod_{i=1}^n \{z_i \in \mathbb{C} \mid z_i \neq 0, z_i \neq z\} \right) = \prod_{i=1}^n \pi_1(\{z_i \in \mathbb{C} \mid z_i \neq 0, z_i \neq z\}) = \prod_{i=1}^n \langle b_{i,0}, b_{i,z} \rangle. \quad (7.11)$$

For  $\lambda \in \text{COMP}$ , we want to define a linear map

$$V \otimes \text{COMP} \ni u \otimes \lambda \mapsto Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)\lambda \in (W_1 \otimes W_2)^* \{x\} [\log x].$$

We first define  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)\lambda \in (W_1 \otimes W_2)^* \{x\} [\log x]$  for  $\lambda$  in a larger subspace of  $(W_1 \otimes W_2)^*$  than  $\text{COMP}$ . Let  $\text{COM}_{P(z)}(W_1, W_2)$  or simply  $\text{COM}$  be the subspace of  $(W_1 \otimes W_2)^*$  consisting of  $\lambda$  satisfying parts 2. and 3. in the  $P(z)$ -compatibility condition. By definition,  $\text{COMP} \subset \text{COM}$ . To define  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)\lambda \in (W_1 \otimes W_2)^* \{x\} [\log x]$  for  $u \in V$  and  $\lambda \in \text{COM}$  is equivalent to define

$$Y_{P(z)}^{(g_1 g_2)^{-1}}(e^{xL(1)}(-x^2)^{-L(0)}u, x^{-1})\lambda = \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, x)\lambda \in (W_1 \otimes W_2)^* \{x\} [\log x].$$

Since  $z_1 = \infty$  is a regular singular point of  $f_1(z_1; u_1, w_1, w_2; \lambda)$ , we know that there exist unique  $a_{i,n,j}(u_1, w_1, w_2; \lambda) \in \mathbb{C}$  and  $r_i \in \mathbb{C}$ , for  $i, j = 0, \dots, K$  and  $n \in \mathbb{N}$ , such that on  $\Omega_{1,0,0}^{(1)}(z) = \Omega_{1,0,0}^{(2)}(z)$  (i.e. the region given by  $|z_1| > |z|$ ,  $|\arg(z_1 - z) - \arg(z_1)| < \frac{\pi}{2}$ ),

$$f_1^e(z_1; u, w_1, w_2; \lambda) = \sum_{i,j=0}^K \sum_{n \in \mathbb{N}} a_{i,n,j}(u, w_1, w_2; \lambda) z_1^{r_i - n} (\log z_1)^j.$$

For  $i, j = 0, \dots, K$ ,  $n \in \mathbb{N}$  and  $u \in V$ , we define  $\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{-r_i + n - 1, j}^o (u)\lambda \in (W_1 \otimes W_2)^*$  by

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{-r_i + n - 1, k}^o (u)\lambda \right) (w_1 \otimes w_2) = a_{i,n,j}(u, w_1, w_2; \lambda)$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . Then we define  $Y_{P(z)}^{(g_1 g_2)^{-1}}(e^{xL(1)}(-x^2)^{L(0)}u, x^{-1})\lambda$  to be

$$\sum_{i,j=0}^N \sum_{n \in \mathbb{N}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{-r_i+n-1,j}^o (u) \lambda x^{r_i-n} (\log x)^j \in (W_1 \otimes W_2)^* \{x\} [\log x],$$

that is,

$$\begin{aligned} & \left( Y_{P(z)}^{(g_1 g_2)^{-1}}(e^{xL(1)}(-x^2)^{-L(0)}u, x^{-1})\lambda \right) (w_1 \otimes w_2) \\ &= \sum_{i,j=0}^N \sum_{n \in \mathbb{N}} a_{i,n,j}(u, w_1, w_2; \lambda) x^{r_i-n} (\log x)^j. \end{aligned} \quad (7.12)$$

By definition,

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, z_1) \lambda \right) (w_1 \otimes w_2)$$

is absolutely convergent on the region  $|z_1| > |z|$  and its sum on  $\Omega_{1,0,0}^{(1)}(z) = \Omega_{1,0,0}^{(2)}(z)$  is equal to  $f_1^e(z_1; u, w_1, w_2; \lambda)$ . For simplicity, let  $\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{m,k}^o (u) = 0$  for either  $m \in \mathbb{C}$ ,  $m \neq -r_i + n - 1$  for  $i = 0, \dots, N$  and  $n \in \mathbb{N}$ , or  $k \in \mathbb{N}$ ,  $k > N$ . Then we have

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, x) = \sum_{m \in \mathbb{C}} \sum_{k \in \mathbb{N}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{m,k}^o (u) x^{-m-1} (\log x)^k.$$

We have the following result:

**Proposition 7.0.5.** *The space COM is invariant under the action of the components of the twisted vertex operators  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)$  for  $u \in V$ .*

*Proof.* We need to show that for  $n_1 \in \mathbb{C}$ ,  $k_1 = 1, \dots, K$ ,  $u_1 \in V$  and  $\lambda \in \text{COM}$ ,

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_1, k_1}^o (u_1) \lambda \in \text{COM}. \quad (7.13)$$

Write  $P = \prod_{1 \leq i < j \leq l} (z_i - z_j)^M$ ,  $Q = \prod_{2 \leq i < j \leq l} (z_i - z_j)^M$ , where  $M \in \mathbb{Z}_+$  is sufficiently large.

Suppose  $\Omega_{0,0,l}^{(1)}$  is nonempty. For any  $l \geq 2$ ,  $u_2, \dots, u_l \in V$ , on the nonempty region

given by  $0 < |z_j| < |z|$ ,  $-\frac{3\pi}{2} < \arg(z_j - z) - \arg(z) < -\frac{\pi}{2}$   $j = 1, \dots, l$ , we know that

$$\begin{aligned}
& P \cdot f_l^e(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda) \\
&= P \cdot \lambda (w_1 \otimes Y^{g_2}(u_1, z_1) \cdots Y^{g_2}(u_l, z_l) w_2) \\
&= (P \cdot \lambda (w_1 \otimes Y^{g_2}(u_1, z_1) Y^{g_2}(u_2, x_2) \cdots Y^{g_2}(u_l, x_l) w_2)) \Big|_{x_i^n = e^{n l_0(z_i)}, \log x_i = l_0(z_i), i=2, \dots, l} \\
&= P \cdot \sum_{n_2, \dots, n_l \in \mathbb{C}} \sum_{k_2, \dots, k_l \in \mathbb{N}} \lambda (w_1 \otimes Y^{g_2}(u_1, z_1) Y_{-n_2-1, k_2}^{g_2}(u_2) \cdots Y_{-n_l-1, k_l}^{g_2}(u_l) w_2) \\
&\quad z_2^{n_2} \log(z_2)^{k_2} \cdots z_l^{n_l} \log(z_l)^{k_l} \\
&= P \cdot \sum_{n_2, \dots, n_l \in \mathbb{C}} \sum_{k_2, \dots, k_l \in \mathbb{N}} f_1^e(z_1; u_1, w_1, Y_{-n_2-1, k_2}^{g_2}(u_2) \cdots Y_{-n_l-1, k_l}^{g_2}(u_l) w_2; \lambda) \\
&\quad z_2^{n_2} \log(z_2)^{k_2} \cdots z_l^{n_l} \log(z_l)^{k_l} \tag{7.14}
\end{aligned}$$

Since for any  $\tilde{z}_1 \in \mathbb{C} - \{0, z\}$ ,  $(z_1, \dots, z_l) = (\tilde{z}_1, 0, \dots, 0)$  is a regular singularity of  $P \cdot f_l$ , we know that the right-hand side of (7.14) is well-defined and absolutely convergent not only on the above region, but on the large region given by  $z_1 \in \mathbb{C} - \{0, z\}$ ,  $|z| > |z_j| > 0$ ,  $j = 2, \dots, l$ . Therefore, we know that on the region given by  $z_1 \in \mathbb{C} - \{0, z\}$ ,  $|z| > |z_j| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_j - z) - \arg(z) < -\frac{\pi}{2}$ ,  $j = 2, \dots, l$ , the left-hand side of (7.14) is equal to the right-hand side of (7.14).

In particular, on the region given by  $z_1 \in \Omega_{1,0,0}^{(1)}(z)$ ,  $|z| > |z_j| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_j - z) - \arg(z) < -\frac{\pi}{2}$ ,  $j = 2, \dots, l$ , we have

$$\begin{aligned}
& P \cdot f_l^e(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda) \\
&= P \cdot \sum_{n_2, \dots, n_l \in \mathbb{C}} \sum_{k_2, \dots, k_l \in \mathbb{N}} f_1^e(z_1; u_1, w_1, Y_{-n_2-1, k_2}^{g_2}(u_2) \cdots Y_{-n_l-1, k_l}^{g_2}(u_l) w_2; \lambda) \\
&\quad z_2^{n_2} \log(z_2)^{k_2} \cdots z_l^{n_l} \log(z_l)^{k_l} \\
&= P \cdot \sum_{n_2, \dots, n_l \in \mathbb{C}} \sum_{k_2, \dots, k_l \in \mathbb{N}} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \lambda \right) (w_1 \otimes Y_{-n_2-1, k_2}^{g_2}(u_2) \cdots Y_{-n_l-1, k_l}^{g_2}(u_l) w_2) \\
&\quad z_2^{n_2} \log(z_2)^{k_2} \cdots z_l^{n_l} \log(z_l)^{k_l} \tag{7.15}
\end{aligned}$$

Since  $(z_1, \dots, z_l) = (\infty, 0, \dots, 0)$  is a regular singularity of  $P \cdot f_l$ , by (7.15) we know that as a multiple series

$$P \cdot \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \lambda \right) (w_1 \otimes Y^{g_2}(u_2, z_2) \cdots Y^{g_2}(u_l, z_l) w_2) \quad (7.16)$$

is absolutely convergent on the region  $|z_1| > |z| > |z_j| > 0$ ,  $j = 2, \dots, l$ . Moreover, it is equal to  $P \cdot f_l^e(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda)$  on the region given by  $z_1 \in \Omega_{1,0,0}^{(1)}(z)$ ,  $|z| > |z_j| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_j - z) - \arg(z) < -\frac{\pi}{2}$ ,  $j = 2, \dots, l$ . This implies that on  $\Omega_{1,0,l-1}^{(1)}(z)$ ,

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \lambda \right) (w_1 \otimes Y^{g_2}(u_2, z_2) \cdots Y^{g_2}(u_l, z_l) w_2)$$

is absolutely convergent to  $f_l^e(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda)$ . Also, since for any  $(\tilde{z}_2, \dots, \tilde{z}_l) \in F_{l-1}(\mathbb{C} - \{0, z\})$ ,  $(z_1, \dots, z_l) = (\infty, \tilde{z}_2, \dots, \tilde{z}_l)$  is a regular singularity of  $P \cdot f_l$ . Therefore, for any  $n_1 \in \mathbb{C}$ ,  $k_1 \in \mathbb{N}$ ,

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_1, k_1}^o (u_1) \lambda \right) (w_1 \otimes Y^{g_2}(u_2, z_2) \cdots Y^{g_2}(u_l, z_l) w_2)$$

is absolutely convergent on the region  $|z_2| > \cdots > |z_l| > 0$  to a function which can be analytically extended to a multivalued analytic function

$$f_{l-1} \left( z_2, \dots, z_l; u_2, \dots, u_l, w_1, w_2; \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_1, k_1}^o (u_1) \lambda \right)$$

on the region  $(z_2, \dots, z_l) \in F_{l-1}(\mathbb{C} - \{0, z\})$ . The fact that this function satisfies Condition 3.0.13 for  $(l-1, 0, \{0, z\})$  can be derived from the fact that

$$f_l(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda)$$

satisfies Condition 3.0.13 for  $(l, 0, \{0, z\})$ .

Now we need to prove that

$$f_{l-1} \left( \frac{z_2 z}{z_{l+1}}, \dots, \frac{z_l z}{z_{l+1}}; u_2, \dots, u_l, w_1, w_2; \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_1, k_1}^o (u_1) \lambda \right) \quad (7.17)$$

satisfies part (d) of Condition 3.0.13 for  $(l-1, 1, \{0\})$ . Suppose we have  $\psi \in \mathcal{S}_{l-1, 1}$ ,  $m \in \{1, \dots, l\}$ , integers  $2 \leq i_2 < \dots < i_m \leq l+1$ , and  $(\tilde{z}_{i_2}, \dots, \tilde{z}_{i_m}) \in F_{m-1}(\mathbb{C} - S)$  (the case  $m = 1$  means there are no such integers  $i_2, \dots, i_m$ ). Without lose any generality, we assume that either  $i_j = j$  for  $j = 2, \dots, m$  or  $i_j = j$  for  $j = 2, \dots, m-1, i_m = l$ . Using the embedding (3.8), we naturally view  $\mathcal{S}_{l-1, 1}$  as a subgroup of  $\mathcal{S}_{l, 1}$ , and view  $\psi$  as an element in either  $\mathcal{S}_{l-1, 1}$  or  $\mathcal{S}_{l, 1}$ . Write  $\psi(z_1, \dots, z_{l+1}) = (\psi_1(z_1, z_{l+1}), \psi_2(z_2, z_{l+1}), \dots, \psi_l(z_l, z_{l+1}), \psi_{l+1}(z_{l+1}))$ .

For the case  $i_j = j$  for  $j = 2, \dots, m$ , we need to prove that

$$\prod_{2 \leq i < j \leq l} (z_i - z_j)^M \cdot f_{l-1} \left( \frac{z \psi_2(z_2, z_{l+1})}{\psi_{l+1}(z_{l+1})}, \dots, \frac{z \psi_l(z_l, z_{l+1})}{\psi_{l+1}(z_{l+1})}; \dots; \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_1, k_1}^o (u_1) \lambda \right) \Big|_{z_i = \tilde{z}_i, i=2, \dots, m} \quad (7.18)$$

satisfies part (b) of Condition 3.0.13 for  $(l-m-2, 1, \{0, \tilde{z}_2, \dots, \tilde{z}_m\})$ . It suffices to prove that

$$\prod_{2 \leq i < j \leq l} (z_i - z_j)^M \cdot f_{l-1} \left( \frac{z \psi_2(z_2, z_{l+1})}{\psi_{l+1}(z_{l+1})}, \dots, \frac{z \psi_l(z_l, z_{l+1})}{\psi_{l+1}(z_{l+1})}; \dots; \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_1, k_1}^o (u_1) \lambda \right) \quad (7.19)$$

satisfies Condition 3.0.13 for  $(l-1, 1, \{0\})$ .

We consider

$$\prod_{2 \leq j \leq l} (z_1 - z_j)^M \cdot \prod_{2 \leq i < j \leq l} (z_i - z_j)^M \cdot f_l \left( \frac{z \psi_1(z_1, z_{l+1})}{\psi_{l+1}(z_{l+1})}, \frac{z \psi_2(z_2, z_{l+1})}{\psi_{l+1}(z_{l+1})}, \dots, \frac{z \psi_l(z_l, z_{l+1})}{\psi_{l+1}(z_{l+1})}; \dots; \lambda \right), \quad (7.20)$$

Since  $\lambda$  is  $P(z)$ -compatible, we know that (7.20) satisfies Condition 3.0.13 for  $(l, 1, \{0\})$ .

Let  $\zeta_1 = \frac{z\psi_1(z_1, z_{l+1})}{\psi_{l+1}(z_{l+1})}$ . Then we have  $z_1 = (\psi^{-1})_1 \left( \frac{\zeta_1 \psi_{l+1}(z_{l+1})}{z}, \psi_{l+1}(z_{l+1}) \right)$ . For any  $\psi \in \mathcal{S}_{l-1,1}$  and fixed  $z_{l+1} \in \mathbb{C}^\times$ , it can be shown that we have

$$\lim_{\zeta_1 \rightarrow \infty} z_1 = \lim_{\zeta_1 \rightarrow \infty} (\psi^{-1})_1 \left( \frac{\zeta_1 \psi_{l+1}(z_{l+1})}{z}, \psi_{l+1}(z_{l+1}) \right) \in \{0, \infty\}, \quad (7.21)$$

or

$$\lim_{\zeta_1 \rightarrow \infty} (z_1 - z_{l+1}) = \lim_{\zeta_1 \rightarrow \infty} (\psi^{-1})_1 \left( \frac{\zeta_1 \psi_{l+1}(z_{l+1})}{z}, \psi_{l+1}(z_{l+1}) \right) - z_{l+1} = 0. \quad (7.22)$$

Notice that  $z_1 = 0, \infty$ , and  $z_1 - z_{l+1} = 0$  are regular singularities of (7.20). Furthermore, they are the singularities mentioned in part (e) of Condition 3.0.13. Take  $z_1 = 0$  as an example, by Appendix A, we know that on the region given by  $|z_1| < |z_i|$ ,  $i = 2, \dots, l+1$ , (7.20) can be written as

$$\sum_{a \in \mathbb{C}, k \in \mathbb{N}} b_{-a-1, k}(z_2, \dots, z_{l+1}) z_1^a \log(z_1)^k = \sum_{n_1 \in \mathbb{C}, k_1 \in \mathbb{N}} b'_{-n_1-1, k_1}(z_2, \dots, z_{l+1}) \zeta_1^{n_1} \log(\zeta_1)^{k_1} \quad (7.23)$$

where  $b_{a, k} = 0$  except for  $a$  is contained in some set  $\cup_{i=1}^l (r_i + \mathbb{N})$  and  $k$  is small enough,  $b'_{n_1, k_1} = 0$  except for  $n_1$  is contained in some set  $\cup_{i=1}^l (r_i - \mathbb{N})$  and  $k_1$  is small enough. We know that  $b'_{n_1, k_1}$  are linear combinations (finite sum!) of  $b_{a, k}$  for  $a \in \mathbb{C}$ ,  $k \in \mathbb{N}$ . The coefficients of these linear combinations are elements in  $\mathbb{C}[z_{l+1}, z_{l+1}^{-1}]$ .

By the fact that (7.20) satisfies Condition 3.0.13 for  $(l, 1, \{0\})$ , in particular, part (e) of Condition 3.0.13, we know that  $b_{a, k}(z_2, \dots, z_{l+1})$ ,  $a \in \mathbb{C}$ ,  $k \in \mathbb{N}$  satisfies Condition 3.0.13 for  $(l-1, 1, \{0\})$ . As a linear combinations of  $b_{a, k}(z_2, \dots, z_{l+1})$  for  $a \in \mathbb{C}$ ,  $k \in \mathbb{N}$ , the function  $b'_{n_1, k_1}(z_2, \dots, z_{l+1})$  also satisfies Condition 3.0.13 for  $(l-1, 1, \{0\})$ .

Notice that (7.19) is equal to  $b'_{n_1, k_1}(z_2, \dots, z_{l+1})$  as multivalued functions.

For the case when  $\Omega_{0,0,l}^{(2)}(z)$  is nonempty, we can use a similar argument as above. ■

**Proposition 7.0.6.** For  $u_1, \dots, u_{m+l} \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $\lambda \in \text{COM}$ , the series

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_m, z_m) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \lambda \right) (w_1 \otimes Y^{g_2}(u_{m+1}, z_{m+1}) \cdots Y^{g_2}(u_{m+l}, z_{m+l}) w_2) \quad (7.24)$$

is absolutely convergent on the region  $|z_1| > \cdots > |z_m| > |z| > |z_{m+1}| > \cdots > |z_{l+m}| > 0$  and its sum is equal to

$$f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}; w_1, w_2; \lambda) \quad (7.25)$$

on  $\Omega_{m,0,l}^{(1)}(z)$  or to

$$f_{m+l}^{b_{m+1,z}^{-1} \cdots b_{m+l,z}^{-1}}(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}; w_1, w_2; \lambda) \quad (7.26)$$

on  $\Omega_{m,0,l}^{(2)}(z)$ . Moreover, we have the following commutativity for  $Y_{P(z)}^{(g_1 g_2)^{-1}}$ : For  $u_1, \dots, u_m \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $\lambda \in \text{COM}$ , the series

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}}(u_1, z_1) \cdots Y_{P(z)}^{(g_1 g_2)^{-1}}(u_m, z_m) \lambda \right) (w_1 \otimes w_2) \quad (7.27)$$

is absolutely convergent on the region  $|z^{-1}| > |z_1| > \cdots > |z_m| > 0$

and for  $\sigma \in S_m$ , the sums of (7.27) and

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}}(u_{\sigma(1)}, z_{\sigma(1)}) \cdots Y_{P(z)}^{(g_1 g_2)^{-1}}(u_{\sigma(m)}, z_{\sigma(l)}) \lambda \right) (w_1 \otimes w_2) \quad (7.28)$$

are analytic extensions of each other. We also have the weak commutativity for

$Y_{P(z)}^{(g_1 g_2)^{-1}}$ : For  $u, v \in V$ , there exists  $M \in \mathbb{Z}_+$  such that

$$(x_1 - x_2)^M Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x_1) Y_{P(z)}^{(g_1 g_2)^{-1}}(v, x_2) = (x_1 - x_2)^M Y_{P(z)}^{(g_1 g_2)^{-1}}(v, x_2) Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x_1). \quad (7.29)$$

*Proof.* This proof uses essentially the same argument as the proof of Lemma 3.0.37.

We use induction on  $m + l$ . For the  $m + l = 1$  case, the result is included in the proof of Proposition 7.0.5. For  $m + l > 1$ , we further use induction on  $m$ . For the  $m = 0$  case, the result is included in the definition of the space COM. For the  $m = 1$  case, the result is included in the proof of Proposition 7.0.5. Now we consider the case  $m > 1$ .

Let  $P = \prod_{1 \leq i < j \leq m+l} (z_i - z_j)^M$ , where  $M \in \mathbb{Z}_+$  is sufficiently large. Suppose  $\Omega_{m,0,l}^{(1)}(z)$  is nonempty. On the nonempty region  $\Omega_{m-1,0,l+1}^{(1)}(z)$ , by using induction hypothesis, we have

$$\begin{aligned} & P \cdot f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda) \\ &= P \cdot \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_{m-1}, z_{m-1}) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) \lambda \right) \\ & \quad (w_1 \otimes Y^{g_2}(u_m, z_m) \cdots Y^{g_2}(u_{m+l}, z_{m+l}) w_2) \\ &= P \cdot \sum_{n_{m+l} \in \mathbb{C}} \sum_{k_{m+l} \in \mathbb{N}} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_{m-1}, z_{m-1}) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) \lambda \right) \\ & \quad (w_1 \otimes Y^{g_2}(u_m, z_m) \cdots (Y^{g_2})_{-n_{m+l}-1, k_{m+l}}(u_{m+l}) w_2) z_{m+l}^{n_{m+l}} \log(z_{m+l})^{k_{m+l}} \\ &= P \cdot \sum_{n_{m+l} \in \mathbb{C}} \sum_{k_{m+l} \in \mathbb{N}} f_{m+l-1}^e \left( z_1, \dots, z_{m+l-1}; u_1, \dots, u_{m+l-1}, w_1, \right. \\ & \quad \left. (Y^{g_2})_{-n_{m+l}-1, k_{m+l}}(u_{m+l}) w_2; \lambda \right) z_{m+l}^{n_{m+l}} \log(z_{m+l})^{k_{m+l}}. \quad (7.30) \end{aligned}$$

Note that  $P \cdot f_{m+l-1}^e \left( z_1, \dots, z_{m+l-1}; u_1, \dots, u_{m+l-1}, w_1, (Y^{g_2})_{-n_{m+l}-1, k_{m+l}}(u_{m+l}) w_2; \lambda \right)$  can be extended to a single valued analytic function on  $F_{m+l-1,0}^1(\mathbb{C} - \{0, z\})$ . For any  $(\tilde{z}_1, \dots, z_{m+l-1}) \in F_{m+l-1}(\mathbb{C} - \{0, z\})$ ,  $(z_1, \dots, z_{m+l}) = (\tilde{z}_1, \dots, z_{m+l-1}, 0)$  is a regular

singularity of  $P \cdot f_{m+l}(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda)$ . Therefore, the right-hand side of (7.30) is well-defined and absolutely convergent to  $P \cdot f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda)$  on the larger region given by  $(z_1, \dots, z_{m+l-1}) \in F_{m+l-1,0}^1(\mathbb{C} - \{0, z\})$ ,  $0 < |z_{m+l}| < |z|$ ,  $|\arg(z_{m+l} - z) - \arg(z_j)| < \frac{\pi}{2}$ .

Using induction hypothesis, we know that on the region  $0 < |z_j| < |z| < |z_i|$ ,  $-\frac{3\pi}{2} < \arg(z_i - z) - \arg(z) < -\frac{\pi}{2}$ ,  $|\arg(z_j - z) - \arg(z_j)| < \frac{\pi}{2}$ ,  $i = 1, \dots, m$ ,  $j = m+1, \dots, m+l-1$ ,

$$\begin{aligned} & P \cdot f_{m+l-1}^e \left( z_1, \dots, z_{m+l-1}; u_1, \dots, u_{m+l-1}, w_1, (Y^{g_2})_{-n_{m+l-1}, k_{m+l}}(u_{m+l})w_2; \lambda \right) \\ = & P \cdot \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_m, z_m) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) \lambda \right) \\ & (w_1 \otimes Y^{g_2}(u_{m+1}, z_{m+1}) \cdots Y^{g_2}(u_{m+l-1}, z_{m+l-1}) (Y^{g_2})_{-n_{m+l-1}, k_{m+l}}(u_{m+l})w_2) \end{aligned} \quad (7.31)$$

Therefore, on the region given by  $0 < |z_j| < |z| < |z_i|$ ,  $-\frac{3\pi}{2} < \arg(z_i - z) - \arg(z) < -\frac{\pi}{2}$ ,  $|\arg(z_j - z) - \arg(z_j)| < \frac{\pi}{2}$ ,  $i = 1, \dots, m$ ,  $j = m+1, \dots, m+l$ .

$$\begin{aligned} & P \cdot f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda) \\ = & P \cdot \sum_{n_{m+l} \in \mathbb{C}} \sum_{k_{m+l} \in \mathbb{N}} f_{m+l-1}^e \left( z_1, \dots, z_{m+l-1}; u_1, \dots, u_{m+l-1}, w_1, \right. \\ & \left. (Y^{g_2})_{-n_{m+l-1}, k_{m+l}}(u_{m+l})w_2; \lambda \right) z_{m+l}^{n_{m+l}} \log(z_{m+l})^{k_{m+l}} \\ = & P \cdot \sum_{n_{m+l} \in \mathbb{C}} \sum_{k_{m+l} \in \mathbb{N}} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_m, z_m) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) \lambda \right) (w_1 \otimes \\ & Y^{g_2}(u_{m+1}, z_{m+1}) \cdots Y^{g_2}(u_{m+l-1}, z_{m+l-1}) (Y^{g_2})_{-n_{m+l-1}, k_{m+l}}(u_{m+l})w_2) \\ & z_{m+l}^{n_{m+l}} \log(z_{m+l})^{k_{m+l}} \\ = & \left[ P \cdot \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_m, z_m) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) \lambda \right) \right] \end{aligned}$$

$$\left. (w_1 \otimes Y^{g_2}(u_{m+1}, z_{m+1}) \cdots Y^{g_2}(u_{m+l-1}, z_{m+l-1}) Y^{g_2}(u_{m+l}, x_{m+l}) w_2) \right] \Big|_{x_{m+l}=z_{m+l}} \quad (7.32)$$

The right-hand side of (7.32) is an iterate series. Since

$$(z_1, \dots, z_{m+l}) = (\overbrace{\infty, \dots, \infty}^m, \overbrace{0, \dots, 0}^l)$$

is a regular singularity of  $P \cdot f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda)$ . We know that the corresponding multiple series of the right-hand side of (7.32) is also absolutely convergent on the same region. Therefore, on the region given by  $0 < |z_j| < |z| < |z_i|$ ,  $-\frac{3\pi}{2} < \arg(z_i - z) - \arg(z) < -\frac{\pi}{2}$ ,  $|\arg(z_j - z) - \arg(z_j)| < \frac{\pi}{2}$ ,  $i = 1, \dots, m$ ,  $j = m+1, \dots, m+l$ , we have

$$\begin{aligned} & P \cdot f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda) \\ &= P \cdot \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_m, z_m) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \lambda \right) \\ & \quad (w_1 \otimes Y^{g_2}(u_{m+1}, z_{m+1}) \cdots Y^{g_2}(u_{m+l}, z_{m+l}) w_2). \end{aligned} \quad (7.33)$$

This would imply that on the region  $(z_1, \dots, z_{m+l}) \in \Omega_{m,0,l}$ , (7.24) is absolutely convergent. Moreover, on the region  $(z_1, \dots, z_{m+l}) \in \Omega_{m,0,l}^{(1)}$ , (7.24) is equal to

$$f_{m+l}^e(z_1, \dots, z_{m+l}; u_1, \dots, u_{m+l}, w_1, w_2; \lambda).$$

Suppose  $\Omega_{m,0,l}^{(2)}$  is nonempty. Replacing  $f_{m+l}^e$ ,  $f_{m+l-1}^e$  by  $f_{m+l}^{b_{m+1,z}^{-1} \cdots b_{m+l,z}^{-1}}$  and  $f_{m+l-1}^{b_{m+1,z}^{-1} \cdots b_{m+l-1,z}^{-1}}$ , and adjusting regions suitably, one would prove the result by the same argument.

This convergence result implies in particular the absolute convergence of (7.27) on the region  $|z^{-1}| > |z_1| > \cdots > |z_l| > 0$ . Using the commutativity for the twisted vertex operators  $Y^{g_2}$ , we see that for  $\sigma \in S_m$ , the sums of (7.27) and (7.28) are

analytic extensions of each other.

For  $u, v \in V$ , and sufficiently large  $M \in \mathbb{Z}_+$ , by the weak commutativity of  $Y_{\tilde{W}_2}^{g_2}$ , we have

$$(z_1 - z_2)^M f_2^e(z_1, z_2; u, v, w_1, w_2; \lambda) = (z_1 - z_2)^M f_2^e(z_2, z_1; v, u, w_1, w_2; \lambda). \quad (7.34)$$

Since the expansion of  $f_2^e(z_1, z_2; u, v, w_1, w_2; \lambda)$  on the region  $\Omega_{2,0,0}^{(1)}(z)$  is

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (v, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, z_1) \lambda \right) (w_1 \otimes w_2),$$

we see that

$$\begin{aligned} & (z_1 - z_2)^M \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (v, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, z_1) \lambda \right) (w_1 \otimes w_2) \\ &= \sum_{i=0}^M \sum_{n \in \mathbb{C}} \sum_{k=0}^{K_1} \sum_{n_1 \in \mathbb{C}} \sum_{k_1=0}^{K_1} \binom{M}{i} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{n_1, k_1} (v) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{n, k} (u) \lambda \right) (w_1 \otimes w_2) \cdot \\ & \quad \cdot e^{(n+M-i) \log z_1} (\log z_1)^k e^{(n_1+i) \log z_2} (\log z_2)^{k_1} \end{aligned} \quad (7.35)$$

must be convergent absolutely to the left-hand side of (7.34) on  $\Omega_{2,0,0}^{(1)}(z)$  (the region given by  $|z_1| > |z_2| > |z|$ ,  $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$ ).

On the other hand,  $(z_1 - z_2)^M f_2^e(z_1, z_2; u, v, w_1, w_2; \lambda)$  is analytic at  $z_1 - z_2 = 0$ . So (7.35) is in fact absolutely convergent to the left-hand side of (7.34) on the region  $|z_1|, |z_2| > |z|$ ,  $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$ . Thus,

$$(z_1 - z_2)^M \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (v, z_2) \lambda \right) (w_1 \otimes w_2) \quad (7.36)$$

is absolutely convergent to the right-hand side of (7.34) also on the region  $|z_1|, |z_2| > |z|$ ,  $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$ . From (7.34), (7.35) and

(7.36), we obtain

$$\begin{aligned} & (z_1 - z_2)^M \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (v, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, z_1) \lambda \right) (w_1 \otimes w_2) \\ &= (z_1 - z_2)^M \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (v, z_2) \lambda \right) (w_1 \otimes w_2) \end{aligned}$$

on the region  $|z_1|, |z_2| > |z|$ ,  $|\arg(z_1 - z) - \arg z_1| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg z_2| < \frac{\pi}{2}$  for  $\lambda \in \text{COM}$ ,  $u_1, v \in V$ ,  $w_1 \in W_1$  and  $w_2 \in W_2$ , which, by the definition of  $\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u, x)$  for  $u \in V$  above, is equivalent to (7.29).  $\blacksquare$

**Proposition 7.0.7.** *For  $u_1, u_2 \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ , and  $\lambda \in \text{COM}$ , the series*

$$\begin{aligned} & \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y(u_1, z_1 - z_2) u_2, z_2) \lambda \right) (w_1 \otimes w_2) \\ &= \left[ \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y(u_1, x_0) u_2, x_2) \lambda \right) (w_1 \otimes w_2) \right] \Big|_{x_0=z_1-z_2, x_2=z_2} \end{aligned} \quad (7.37)$$

is absolutely convergent on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ , and its sum is equal to  $f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$  on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ , and  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ .

*Proof.* On the region given by  $0 < |z_1 - z_2| < |z_2| < |z_1|$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ , by duality of  $Y_{W_2}^{g_2}$ , as elements in  $\overline{W_2}$ , we have

$$Y_{W_2}^{g_2}(u_1, z_1) Y_{W_2}^{g_2}(u_2, z_2) w_2 = Y_{W_2}^{g_2}(Y_V(u_1, z_1 - z_2) u_2, z_2) w_2. \quad (7.38)$$

Let  $\tilde{w}_2 \in \overline{W_2}$  be the element (7.38). We know that for any  $n \in \mathbb{C}$ ,

$$\pi_n(\tilde{w}_2) = \sum_{m \in \mathbb{Z}} (Y_{W_2})_{-n-1+\Delta'+m} (Y_{-m-1}(u_1) u_2) w_2 \cdot (z_1 - z_2)^m e^{(-\Delta'-m+n)l_0(z_2)},$$

where  $\Delta' = \text{wt } u_1 + \text{wt } u_1 + \text{wt } w_2$ . Therefore, on the region given by  $0 < |z_1 - z_2| <$

$|z_2| < |z|$  and  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ , we have

$$\begin{aligned}
& \sum_{n \in \mathbb{C}} \left( \sum_{m \in \mathbb{Z}} (Y_{W_2})_{-n-1+\Delta'+m} (Y_{-m-1}(u_1)u_2) w_2 \cdot e^{(n-\Delta')l_p(z)} \left( \frac{z_1 - z_2}{z_2} \right)^m \right) e^{(\Delta'-n)l_0\left(\frac{z}{z_2}\right)} \\
&= \sum_{n \in \mathbb{C}} \lambda(w_1 \otimes \pi_n Y_{W_2}^{g_2} (Y_V(u_1, z_1 - z_2)u_2, z_2) w_2) = \sum_{n \in \mathbb{C}} \lambda(w_1 \otimes \pi_n \tilde{w}_2) \\
&= \sum_{n \in \mathbb{C}} \lambda(w_1 \otimes \pi_n Y_{W_2}^{g_2}(u_1, z_1) Y_{W_2}^{g_2}(u_2, z_2) w_2), \tag{7.39}
\end{aligned}$$

where  $p = p(z, z_2)$  is the unique integer such that  $l_0\left(\frac{z}{z_2}\right) = l_p(z) - l_0(z_2)$ . We know that  $p$  is locally constant. Note that the right-hand side of (7.39), as a summation over  $n \in \mathbb{C}$ , is absolutely convergent on the region given by  $0 < |z_2| < |z_1|$  to  $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$ , because

$$\lambda(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1) Y_{W_2}^{g_2}(u_2, z_2) w_2),$$

as a multiple series on  $z_1$  and  $z_2$ , is absolutely convergent on the region given by  $0 < |z_2| < |z_1|$  to  $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$ . Therefore, we know that the left-hand side of (7.39), as an iterate series, is absolutely convergent to  $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$ , on the region given by  $0 < |z_1 - z_2| < |z_2| < |z_1|$ , and  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ .

Let  $(a_1, a_2) = \left(\frac{z}{z_2}, \frac{z_1 - z_2}{z_2}\right)$ , and  $g(a_1, a_2) = \left(\frac{za_2}{a_1}\right)^M f_2^e\left(\frac{z(a_2+1)}{a_1}, \frac{z}{a_1}; u_1, u_2, w_1, w_2; \lambda\right) = (z_1 - z_2)^M f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$ . Since  $\lim_{(a_1, a_2) \rightarrow (\infty, 0)} (z_1, z_2) = (0, 0)$ , and  $(z_1, z_2) = (0, 0)$  is a regular singularity of  $(z_1 - z_2)^M f_2^e(z_1, z_2)$ , by Appendix A, we know that  $(a_1, a_2) = (\infty, 0)$  is a regular singularity of  $g(z_1, z_2)$ . Therefore, on the region given by  $|a_1| > 2, 0 < |a_2| < 1$ , or equivalently  $0 < |z_1 - z_2| < |z_2| < \frac{1}{2}|z|$ , we have a corresponding expansion, which has the same form as the corresponding multiple series of the iterate series, the left-hand side of (7.39). Therefore we know that

$$\lambda(w_1 \otimes Y_{W_2}^{g_2} (Y_V(u_1, z_1 - z_2)u_2, z_2) w_2) \tag{7.40}$$

is absolutely convergent to  $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$  on the region  $0 < |z_1 - z_2| < |z_2| < \frac{1}{2}|z|$ , which can be enlarged to  $0 < |z_1 - z_2| < |z_2|$ ,  $0 < |z_1 - z_2| + |z_2| < |z|$ . Moreover, by (7.38) and (7.39), we know that (7.40) is equal to  $f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$  on the region given by  $0 < |z_1 - z_2| < |z_2|$ ,  $0 < |z_1 - z_2| + |z_2| < |z|$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $-\frac{3\pi}{2} < \arg(z_i - z) - \arg(z) < -\frac{\pi}{2}$ ,  $i = 1, 2$ . And (7.40) is equal to  $f_2^{b_1^{-1}, z b_2^{-1}}(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$  on the region given by  $0 < |z_1 - z_2| < |z_2|$ ,  $0 < |z_1 - z_2| + |z_2| < |z|$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $\frac{\pi}{2} < \arg(z_i - z) - \arg(z) < \frac{3\pi}{2}$ ,  $i = 1, 2$ .

Suppose the region  $\Omega_{0,0,2}^{(1)}(z)$  is nonempty. On the region given by  $0 < |z_1 - z_2| < |z_2|$ ,  $0 < |z_1 - z_2| + |z_2| < |z|$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $(z_1, z_2) \in \Omega_{0,0,2}^{(1)}(z)$ ,

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} f_1^e(z_2; (u_1)_n u_2, w_1, w_2; \lambda) (z_1 - z_2)^{-n-1} \\
&= \sum_{n \in \mathbb{Z}} \lambda (w_1 \otimes Y((u_1)_n u_2, z_2) w_2) (z_1 - z_2)^{-n-1} \\
&= \lambda (w_1 \otimes Y_{W_2}^{g_2} (Y_V(u_1, z_1 - z_2) u_2, z_2) w_2) = \lambda (w_1 \otimes Y_{W_2}^{g_2} (u_1, z_1) Y_{W_2}^{g_2} (u_2, z_2) w_2) \\
&= f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda). \tag{7.41}
\end{aligned}$$

Since for any fixed  $\tilde{z}_2 \in \mathbb{C} - \{0, z\}$ ,  $f_2(z_1, \tilde{z}_2; u_1, u_2, w_1, w_2; \lambda)$  is analytic on the region given by  $0 < |z_1 - \tilde{z}_2| < \min\{|z - \tilde{z}_2|, |\tilde{z}_2|\}$ , with a regular singularity  $z_1 - \tilde{z}_2 = 0$ . Therefore, the left-hand side of (7.41) is absolutely convergent on the region given by  $0 < |z_1 - z_2| < \min\{|z - z_2|, |z_2|\}$ . By the analyticity of both sides of (7.41), we know that

$$\sum_{n \in \mathbb{Z}} f_1^e(z_2; (u_1)_n u_2, w_1, w_2; \lambda) (z_1 - z_2)^{-n-1} = f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$$

holds on the region given by  $0 < |z_1 - z_2| < \min\{|z - z_2|, |z_2|\}$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ . In particular, on the sub-region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ ,  $|\arg(z_1) - \arg(z_2)| <$

$\frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ , we have

$$\begin{aligned}
& \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y_V(u_1, z_1 - z_2) u_2, z_2) (\lambda) (w_1 \otimes w_2) \\
&= \sum_{n \in \mathbb{Z}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o ((u_1)_n u_2, z_2) (\lambda) (w_1 \otimes w_2) (z_1 - z_2)^{-n-1} \\
&= \sum_{n \in \mathbb{Z}} f_1^e(z_2; (u_1)_n u_2) (z_1 - z_2)^{-n-1} = f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda). \tag{7.42}
\end{aligned}$$

Suppose the region  $\Omega_{0,0,2}^{(2)}(z)$  is nonempty. On the region given by  $0 < |z_1 - z_2| < |z_2|$ ,  $0 < |z_1 - z_2| + |z_2| < |z|$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $(z_1, z_2) \in \Omega_{0,0,2}^{(2)}(z)$ ,

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} f_1^{b_{2,z}^{-1}}(z_2; (u_1)_n u_2, w_1, w_2; \lambda) (z_1 - z_2)^{-n-1} \\
&= \sum_{n \in \mathbb{Z}} \lambda (w_1 \otimes Y((u_1)_n u_2, z_2) w_2) (z_1 - z_2)^{-n-1} \\
&= \lambda (w_1 \otimes Y_{W_2}^{g_2} (Y_V(u_1, z_1 - z_2) u_2, z_2) w_2) \\
&= \lambda (w_1 \otimes Y_{W_2}^{g_2} (u_1, z_1) Y_{W_2}^{g_2} (u_2, z_2) w_2) \\
&= f_2^{b_{1,z}^{-1} b_{2,z}^{-1}}(z_1, z_2; u_1, u_2, w_1, w_2; \lambda). \tag{7.43}
\end{aligned}$$

Similar as the  $\Omega_{0,0,2}^{(1)}(z) \neq \emptyset$  case, we know that the left-hand side of (7.43) is absolutely convergent on the region given by  $0 < |z_1 - z_2| < \min\{|z - z_2|, |z_2|\}$ . Also,

$$\sum_{n \in \mathbb{Z}} f_1^{b_{2,z}^{-1}}(z_2; (u_1)_n u_2, w_1, w_2; \lambda) (z_1 - z_2)^{-n-1} = f_2^{b_{1,z}^{-1} b_{2,z}^{-1}}(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$$

holds on the region  $U$  given by  $0 < |z_1 - z_2| < \min\{|z - z_2|, |z_2|\}$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ . For any  $(\tilde{z}_1, \tilde{z}_2) \in U$ , we can always find a path contained in  $U$ , starting and ending at  $(z_1, z_2)$ , and representing  $b_{1,z} b_{2,z}$ . We define a path  $(z_1(t), z_2(t))$  in  $U$  starting and ending at  $(z_1, z_2)$  as follows. Assume  $\arg \tilde{z}_1 < \arg \tilde{z}_2$ . For  $t \in [0, 1/3]$ , let  $(z_1(t), z_2(t)) = (e^{i((1-3t)\arg \tilde{z}_1 + 3t\arg \tilde{z}_2)} \tilde{z}_1, \tilde{z}_2)$ . For  $t \in [1/3, 2/3]$ , let  $(z_1(t), z_2(t)) = (z + e^{2\pi i(3t-1)}(z_1(1/3) - z), z + e^{2\pi i(3t-1)}(\tilde{z}_2 - z))$ . For  $t \in [1/3, 2/3]$ , reverse the first

one third of the path. We know that  $\gamma = (z_1(t), z_2(t))$  satisfying the requirement we want. Therefore, on the region  $(z_1, z_2) \in U$ , we have

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} f_1^e(z_2; (u_1)_n u_2, w_1, w_2; \lambda) (z_1 - z_2)^{-n-1} \\
&= [\gamma] \cdot \left( \sum_{n \in \mathbb{Z}} f_1^{b_{2,z}^{-1}}(z_2; (u_1)_n u_2, w_1, w_2; \lambda) (z_1 - z_2)^{-n-1} \right) \\
&= [\gamma] \cdot \left( f_2^{b_{1,z}^{-1} b_{2,z}^{-1}}(z_1, z_2; u_1, u_2, w_1, w_2; \lambda) \right) \\
&= f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda).
\end{aligned}$$

Same as before, we know that (7.42) holds on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ .  $\square$

**Proposition 7.0.8.** *On the space COM, for any  $v \in V$ , we have the  $L(-1)$ -derivative property*

$$Y_{P(z)}^{(g_1 g_2)^{-1}}(L(-1)v, x) = \frac{d}{dx} Y_{P(z)}^{(g_1 g_2)^{-1}}(v, x).$$

*Proof.* Let  $\lambda \in \text{COM}$ . On the region  $z_1 \in \Omega_{0,0,1}^{(1)}(z)$ , for any  $u_1 \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , we have

$$\begin{aligned}
& f_1^e(z_1; L(-1)u_1, w_1, w_2; \lambda) \\
&= \lambda(w_1 \otimes Y_{W_2}^{g_2}(L(-1)u_1, z_1)w_2) = \left( \lambda(w_1 \otimes Y_{W_2}^{g_2}(L(-1)u_1, x_1)w_2) \right) \Big|_{x_1=z_1} \\
&= \left( \frac{d}{dx_1} \lambda(w_1 \otimes Y_{W_2}^{g_2}(u_1, x_1)w_2) \right) \Big|_{x_1=z_1} \\
&= \frac{d}{dz_1} \lambda(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) \\
&= \frac{d}{dz_1} f_1^e(z_1; u_1, w_1, w_2; \lambda)
\end{aligned} \tag{7.44}$$

Therefore, by the holomorphicity, we know that on the region given by  $z_1 \in \mathbb{C} - \{0, z\}$ ,

$$f_1^e(z_1; L(-1)u_1, w_1, w_2; \lambda) = \frac{d}{dz_1} f_1^e(z_1; u_1, w_1, w_2; \lambda)$$

always holds. In particular, on the region given by  $z_1 \in \Omega_{1,0,0}^{(1)}(z)$ , we have

$$\begin{aligned}
& \left( \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (L(-1)u_1, x_1) \lambda \right) (w_1 \otimes w_2) \right) \Big|_{x_1=z_1} \\
&= \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (L(-1)u_1, z_1) \lambda \right) (w_1 \otimes w_2) = f_1^e(z_1; L(-1)u_1, w_1, w_2; \lambda) \\
&= \frac{d}{d z_1} f_1^e(z_1; u_1, w_1, w_2; \lambda) \\
&= \frac{d}{d z_1} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \lambda \right) (w_1 \otimes w_2) \\
&= \left( \frac{d}{d x_1} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1) \lambda \right) (w_1 \otimes w_2) \right) \Big|_{x_1=z_1}. \tag{7.45}
\end{aligned}$$

Therefore, we know that, for any  $u_1 \in V$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ ,

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (L(-1)u_1, x_1) \lambda \right) (w_1 \otimes w_2) = \frac{d}{d x_1} \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1) \lambda \right) (w_1 \otimes w_2).$$

Therefore we have

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (L(-1)u_1, x_1) = \frac{d}{d x_1} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1),$$

which is equivalent to the  $L(-1)$ -derivative property. ■

For any  $g \in \text{Aut}(V)$ , We define the follow vector space:

$$\begin{aligned}
\text{COM}_g &= \text{COM}_{g, P(z)}(W_1, W_2) = \\
& \left\{ \lambda \in \text{COM} \left| \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^1 (gv, x) (\lambda) = \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^0 (v, x) (\lambda), \text{ for any } v \in V. \right. \right\} \tag{7.46}
\end{aligned}$$

**Proposition 7.0.9.** *The space  $\text{COM}_g$  is invariant under the action of the components of the twisted vertex operators  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)$  for  $u \in V$ .*

*Proof.* Let  $\lambda \in \text{COM}_g$ ,  $u_1, u_2 \in V$ ,  $w_1 \in W$  and  $w_2 \in W_2$ . By Proposition 7.0.5, we know that  $\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n,k}^o (u_1) \lambda \in \text{COM}$  for any  $n \in \mathbb{C}$ ,  $k \in \mathbb{N}$ . By the weak

commutativity in Proposition 7.0.6, we know that there exists  $M \in \mathbb{Z}_+$ , such that

$$\begin{aligned}
& (x_1 - x_2)^M \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (g^{-1} u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda) \\
&= (x_1 - x_2)^M \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (g^{-1} u_1, x_1)(\lambda) \\
&= (x_1 - x_2)^M \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2) \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{-1} (u_1, x_1)(\lambda) \\
&= (x_1 - x_2)^M \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{-1} (u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda). \tag{7.47}
\end{aligned}$$

By Proposition 7.0.6, both

$$\begin{aligned}
& \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (g^{-1} u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2), \quad \text{and} \\
& \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{-1} (u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2)
\end{aligned}$$

are absolutely convergent on the region  $(z_2, z_1) \in \Omega_{2,0,0}(z)$ . Therefore, by Proposition 7.0.6 and (7.47), we know that on the region given by  $(z_2, z_1) \in \Omega_{2,0,0}(z)$ , we have

$$\begin{aligned}
& (z_1 - z_2)^M \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (g^{-1} u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2) \\
&= (z_1 - z_2)^M \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{-1} (u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2) \tag{7.48}
\end{aligned}$$

Multiplying  $(z_1 - z_2)^{-M}$ , which is an absolutely convergent series on  $(z_2, z_1) \in \Omega_{2,0,0}(z)$ , on the both sides of (7.48), we know that

$$\begin{aligned}
& \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (g^{-1} u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2) \\
&= \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{-1} (u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2),
\end{aligned}$$

This implies the formal variable version

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (g^{-1} u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda)$$

$$= \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{-1} (u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda), \quad (7.49)$$

which is equivalent to

$$\begin{aligned} & \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^1 (g u_1, x_1)(\lambda) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda) \\ &= \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^0 (u_1, x_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda). \end{aligned}$$

■

**Proposition 7.0.10.** *Suppose  $\lambda \in \text{COM}_g = \text{COM}_{g, P(z)}(W_1, W_2)$ . Then, for any  $u_1 \in V^{(g_2, g)}$ ,  $u_2 \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ , we have the following Jacobi identity,*

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1)(\lambda)(w_1 \otimes w_2) \\ & - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, x_2)(\lambda)(w_1 \otimes w_2) \\ & = x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y_V(u_1, x_0) u_2, x_2)(\lambda)(w_1 \otimes w_2). \end{aligned} \quad (7.50)$$

*In the sense that the coefficients of each formal monomial in both sides of (7.50) is absolutely convergent.*

*Proof.* Let  $\lambda \in \text{COM}_{g, P(z)}(W_1, W_2)$ . In the following of this proof, we write  $f_2^e(z_1, z_2)$  and  $f_1^e(z_1; v)$  as short for  $f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda)$  and  $f_1^e(z_1; v, w_1, w_2; \lambda)$ , respectively.

Since  $g_2 u_1 = u_1$ , by the equivariant property, we know that

$$Y_{W_2}^{g_2}(u_1, x) \in \text{End}(W_2)[[x, x^{-1}]].$$

Similarly, by  $\lambda \in \text{COM}_g$  and the fact  $g u_1 = u_1$ , we know that  $\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1)(\lambda) \in$

COM[[ $x^{\pm 1}$ ]]. For any  $z_1, z_2 \in \mathbb{C}$  with  $(z_2, z_1) \in \Omega_{1,0,1}^{(1)}(z)$ , we have

$$f_2^e(z_1, z_2) = \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) \quad (7.51)$$

Therefore, for any  $z_2 \in \mathbb{C}$  with  $|z_2| > |z|$ , we know that there is no multivaluedness for  $f_2^e(z_1, z_2)$  when  $z_1$  goes around 0.

For any  $z_1, z_2 \in \mathbb{C}$  with  $(z_1, z_2) \in \Omega_{2,0,0}^{(1)}(z)$ , we have

$$f_2^e(z_1, z_2) = \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1)(\lambda)(w_1 \otimes w_2). \quad (7.52)$$

Therefore, for any  $z_2 \in \mathbb{C}$  with  $|z_2| > |z|$ , we know that there is no multivaluedness for  $f_2^e(z_1, z_2)$  when  $z_1$  goes around  $\infty$ .

Similar as (7.52), by Proposition 7.0.6, for any  $z_1, z_2 \in \mathbb{C}$  with  $(z_2, z_1) \in \Omega_{2,0,0}^{(1)}(z)$ , we have

$$f_2^e(z_1, z_2) = \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2)(\lambda)(w_1 \otimes w_2). \quad (7.53)$$

After fixing any  $z_2 \in \mathbb{C}$  with  $|z_2| > |z|$ ,  $f_2^e(z_1, z_2)$  is a function on  $z_1 \in \hat{\mathbb{C}}$ ,  $z_1 \neq 0, z, z_2, \infty$ , where  $z_1 = 0, z_2, \infty$  are poles of  $f_2^e(z_1, z_2)$ . The only possible branch point is  $z_1 = z$ , which implies that there is no multivaluedness for  $f_2^e(z_1, z_2)$  when  $z_1$  goes around  $z$ . Since  $z_1 = z$  is a regular singularity,  $z_1 = z$  is a pole of  $f_2^e(z_1, z_2)$ . Thus, for any fixed  $z_2 \in \mathbb{C}$  with  $|z_2| > |z|$ , as a function on  $z_1$ ,  $f_2^e(z_1, z_2)$  is a single valued meromorphic function on  $\hat{\mathbb{C}}$  with poles at  $z_1 = 0, z, z_2, \infty$ . In other words, there exists  $g_{z_2}(z_1) \in \mathbb{C}[z_1]$ ,  $l_1, l_2, l_3 \in \mathbb{Z}$ , such that

$$f_2^e(z_1, z_2) = \frac{g_{z_2}(z_1)}{z_1^{l_1}(z_1 - z)^{l_2}(z_1 - z_2)^{l_3}}. \quad (7.54)$$

Notice that by taking the coefficient of  $x_0^{-k-1}x_1^{-j-1}$ , (7.50) can be implied by the fact

that on some open region of  $z_2$  contained in  $|z_2| > |z|$ ,

$$\begin{aligned}
& \operatorname{Res}_{x_1} x_1^j (x_1 - z_2)^k \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, x_1) (\lambda) (w_1 \otimes w_2) \\
& - \operatorname{Res}_{x_1} x_1^j (-z_2 + x_1)^k \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) (\lambda) (w_1 \otimes w_2) \\
& = \operatorname{Res}_{x_0} x_0^k (z_2 + x_0)^j \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(Y_V(u_1, x_0) u_2, z_2) (\lambda) (w_1 \otimes w_2), \text{ for all } j, k \in \mathbb{Z}.
\end{aligned} \tag{7.55}$$

By (7.52), we know that

$$\begin{aligned}
& \operatorname{Res}_{x_1} x_1^j (x_1 - z_2)^k \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, x_1) (\lambda) (w_1 \otimes w_2) \\
& = \frac{1}{2\pi i} \int_{\Gamma_1} z_1^j (z_1 - z_2)^k f_2^e(z_1, z_2) \, d z_1,
\end{aligned} \tag{7.56}$$

where  $\Gamma_1$  is clockwise going around the circle of radius  $|z_2| + 2$  centered at 0 (a loop with only  $\infty$  inside and  $0, z, z_2$  outside). Similarly, by (7.53), we have

$$\begin{aligned}
& - \operatorname{Res}_{x_1} x_1^j (-z_2 + x_1)^k \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, x_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) (\lambda) (w_1 \otimes w_2) \\
& = \frac{1}{2\pi i} \int_{\Gamma_2} z_1^j (-z_2 - z_1)^k f_2^e(z_1, z_2) \, d z_1,
\end{aligned} \tag{7.57}$$

where  $\Gamma_2$  is clockwise going around the circle of radius  $\frac{1}{2}(|z| + |z_2|)$  centered at 0 (a loop inside the annulus  $|z| < |z_1| < |z_2|$ ).

Now we consider the right-hand side. By Proposition 7.0.7, we know that

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(Y(u_1, z_1 - z_2) u_2, z_2) \lambda \right) (w_1 \otimes w_2) = f_2^e(z_1, z_2) \tag{7.58}$$

on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ , and  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ . Since there is no multivaluedness around  $z_1 = z$  and  $z_1 = 0$  for  $f_2(z_1, z_2)$ , we know that (7.58) holds on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ ,  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ .

Therefore, for any  $z_2$  with  $|z_2| > |z|$ , and  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ , we have

$$\begin{aligned}
& \text{Res}_{x_0} x_0^k (z_2 + x_0)^j \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y_V(u_1, x_0) u_2, z_2) (\lambda) (w_1 \otimes w_2) \\
&= \frac{1}{2\pi i} \int_{\Gamma_3} (z_1 - z_2)^k (z_2 + z_1 - z_2)^j \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y_V(u_1, z_1 - z_2) u_2, z_2) (\lambda) (w_1 \otimes w_2) \, dz_1 \\
&= \frac{1}{2\pi i} \int_{\Gamma_3} (z_1 - z_2)^k (z_1)^j f_2^e(z_1, z_2) \, dz_1, \tag{7.59}
\end{aligned}$$

where  $\Gamma_3$  is counterclockwise going around the circle with radius  $\frac{1}{2}(|z_2| - |z|)$  centered at  $z_2$ .

By (7.56), (7.57), (7.59), and Cauchy formula, we know that (7.55) holds on the region given by  $|z_2| > |z|$ , and  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ , which implies that (7.50) holds.  $\blacksquare$

For any  $n \in \mathbb{Z}$ , we define the linear operator  $L(n)'_{P(z)}$  on COM as the coefficient of  $x^{-n-2} \log(x)^0$  in  $Y_{P(z)}^{(g_1 g_2)^{-1}}(\omega, x)$ .

**Remark 7.0.11.** *It can be proved that for any  $u \in V^{\langle g_1, g_2 \rangle}$ , the operator  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)$  is the same as the operator  $Y'_{P(z)}(u, x)$  defined in definition 5.3 in [HLZ4] using the approach of formal calculus where we regard  $W_i$ ,  $i = 1, 2$ , as untwisted generalized  $V^{\langle g_1, g_2 \rangle}$ -modules. Note that  $\omega \in V^{\text{Aut}(V)} \subset V^{\langle g_1, g_2 \rangle}$ . Therefore, we know that there is no logarithm term in  $Y_{P(z)}^{(g_1 g_2)^{-1}}(\omega, x)$ .*

By letting  $u_1$  in (7.50) be  $\omega$  and taking  $\text{Res}_{x_0}$  on (7.50), we can prove the following result.

**Corollary 7.0.12.** *For any  $g \in \text{Aut}(V)$ , on the space  $\text{COM}_g$ , the Virasoro-bracket relations and the Virasoro module structure hold. In other words, for any  $v \in V$ , on  $\text{COM}_g$ , we have*

$$\left[ L'_{P(z)}(m), Y_{P(z)}^{(g_1 g_2)^{-1}}(v, x) \right] = \sum_{k \in \mathbb{N}} \binom{m+1}{k} x^{m-k+1} Y_{P(z)}^{(g_1 g_2)^{-1}}(L(k-1)v, x), \quad m \in \mathbb{Z}. \tag{7.60}$$

$$[L'_{P(z)}(m), L'_{P(z)}(n)] = (m - n)L'_{P(z)}(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} c_V, \quad m, n \in \mathbb{Z}. \quad (7.61)$$

Since  $\text{COMP} \subset \text{COM}$ , by Proposition 7.0.5,  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)\lambda$  for  $u \in V$  and  $\lambda \in \text{COMP}$  is in  $\text{COM}\{x\}[\log x]$ . We now prove the following stronger result:

**Proposition 7.0.13.** *The space COMP is invariant under the action of the components of the twisted vertex operators  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)$  for  $u \in V$ .*

*Proof.* We only need to show that for any  $\lambda \in \text{COMP}$ ,  $u_1, u_2 \in V$ ,  $n_2 \in \mathbb{C}$ ,  $k_2 \in \mathbb{N}$ , the series

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_2, k_2}^o (u_2) \lambda \right) (Y^{g_1}(u_1, z_1 - z) w_1 \otimes w_2) \quad (7.62)$$

is absolutely convergent on the region  $\Omega_{0,1,0}(z)$  (i.e.  $0 < |z_1 - z| < |z|$ ), and is equal to

$$f_1^e \left( z_1; u_1, w_1, w_2; \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_2, k_2}^o (u_2) \lambda \right) \quad (7.63)$$

on the region given by  $z_1 \in \Omega_{0,1,0}^{(1)}(z)$ .

Using a similar argument as the proof of Proposition 7.0.6, we can prove that

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)_{n_2, k_2}^o (u_2, z_2) \lambda \right) (Y^{g_1}(u_1, z_1 - z) w_1 \otimes w_2) \quad (7.64)$$

is absolutely convergent on the region  $\Omega_{1,1,0}(z)$  (i.e. the region given by  $0 < |z_1 - z| + |z| < |z_2|$ ). Moreover, on the region  $\Omega_{1,1,0}^{(1)}(z) = \Omega_{1,1,0}^{(2)}(z)$ , it is equal to

$$f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda).$$

The absolute convergence of (7.64) implies the absolute convergence of (7.62).

On the region given by  $(z_2, z_1) \in \Omega_{1,1,0}^{(1)}(z) \cap \Omega_{1,0,1}^{(1)}(z)$ , we have

$$\begin{aligned}
& \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2) \lambda \right) (Y^{g_1}(u_1, z_1 - z) w_1 \otimes w_2) \\
&= f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda) \\
&= \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2) \lambda \right) (w_1 \otimes Y^{g_2}(u_1, z_1) w_2). \tag{7.65}
\end{aligned}$$

On the region given by  $(z_2, z_1) \in \Omega_{1,1,0}^{(1)}(z) \cap \Omega_{1,0,1}^{(2)}(z)$ , we have

$$\begin{aligned}
& \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2) \lambda \right) (Y^{g_1}(u_1, z_1 - z) w_1 \otimes w_2) \\
&= f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \lambda) \\
&= b_{1,z} \cdot \left( f_2^{b_{1,z}^{-1}}(z_1, z_2; u_1, u_2, w_1, w_2; \lambda) \right) \\
&= b_{1,z} \cdot \left[ \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2) \lambda \right) (w_1 \otimes Y^{g_2}(u_1, z_1) w_2) \right]. \tag{7.66}
\end{aligned}$$

Using (7.65) and (7.66), one can show that (7.62) is equal (7.63) on the region given by  $z_1 \in \Omega_{0,1,0}^{(1)}(z)$ .  $\blacksquare$

In general, one can use an argument similar to the proof of Proposition 7.0.6 and Proposition 7.0.13 to prove the following proposition.

**Proposition 7.0.14.** *Let  $\lambda \in \text{COMP}_{P(z)}(W_1, W_2)$ . For any  $m, k, l \in \mathbb{N}$ , the multiple series*

$$\begin{aligned}
& \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_m, x_m) \cdots \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, x_1) (\lambda) \right) \\
& \left( Y^{g_1}(u_{m+1}, x_{m+1} - z) \cdots Y^{g_1}(u_{m+k}, x_{m+k} - z) w_1 \otimes \right. \\
& \quad \left. Y^{g_2}(u_{m+k+1}, x_{m+k+1}) \cdots Y^{g_2}(u_{m+k+l}, x_{m+k+l}) w_2 \right) \tag{7.67}
\end{aligned}$$

is absolutely convergent on the region  $\Omega_{m,k,l}(z)$ . Moreover, on the region  $\Omega_{m,k,l}^{(1)}(z)$ , it

is convergent to

$$\tilde{f}_{m+k+l}^e(z_1, \dots, z_{m+k+l}; u_1, \dots, u_{m+k+l}, w_1, w_2; \lambda).$$

On the region  $\Omega_{m,k,l}^{(2)}(z)$ , it is convergent to

$$\tilde{f}_{m+k+l}^{\prod_{i=k+m+1}^{k+m+l} b_{i,z}^{-1}}(z_1, \dots, z_{m+k+l}; u_1, \dots, u_{m+k+l}, w_1, w_2; \lambda).$$

■

Let  $F\langle x_1, x_2 \rangle$  be the free group with generators  $x_1, x_2$ . For any  $i = 1, \dots, n$ , define the group homomorphism  $\varphi_i : F\langle x_1, x_2 \rangle \rightarrow$  the group (7.11) by  $\varphi_i(x_1) = b_{i,z}$ ,  $\varphi_i(x_2) = b_{i,z}^{-1} b_{i,0} b_{i,z}$ . Define the group homomorphism  $\psi : F\langle x_1, x_2 \rangle \rightarrow \langle g_1, g_2 \rangle$  by  $\psi(x_k) = g_k$ ,  $k = 1, 2$ .

Similar as Lemma 3.0.34 and Corollary 3.0.35, we can prove the following corollary.

**Corollary 7.0.15.** *Let  $\lambda \in \text{COMP}_{P(z)}((W_1 \otimes W_2)^*)$ , and  $\{(f_n, f_n^e)\}_{n \in \mathbb{N}}$  be the multi-valued functions with preferred branches in the definition of  $P(z)$ -compatibility condition of  $\lambda$ . For any  $a \in$  the group (7.11), and  $y_1, \dots, y_n \in F\langle x_1, x_2 \rangle$ , we have*

$$\begin{aligned} & f_n^a(z_1, \dots, z_n; \psi(y_1)u_1, \dots, \psi(y_n)u_n, w_1, w_2; \lambda) \\ &= f_n^{(\prod_{i=1}^n \varphi_i(y_i)^{-1})^a}(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2, w_3') \end{aligned}$$

**Corollary 7.0.16.**  $\text{COMP} \subset \text{COM}_{(g_1 g_2)^{-1}}$ .

*Proof.* Let  $\lambda \in \text{COMP}$ . We shall prove the following equivalent form:

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{p+1} ((g_1 g_2)u, x)(\lambda) = \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^p (u, x)(\lambda), \text{ for any } u \in V, p \in \mathbb{Z}. \quad (7.68)$$

It is sufficient to prove that for any  $w_1 \in W_1$ ,  $w_2 \in W_2$ , on the region given by

$$|z_1| > |z|, |\arg(z_1 - z) - \arg(z_1)| < \frac{\pi}{2},$$

$$\left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{p+1} ((g_1 g_2)u, z_1)(\lambda)(w_1 \otimes w_2) = \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^p (u, z_1)(\lambda)(w_1 \otimes w_2).$$

By Lemma 7.0.15, we know that on the region  $|z_1| > |z|, |\arg(z_1 - z) - \arg(z_1)| < \frac{\pi}{2}$ ,

$$\begin{aligned} & \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^{p+1} ((g_1 g_2)u, z_1)(\lambda)(w_1 \otimes w_2) \\ &= f_1^{(b_{1,0} b_{1,z})^{p+1}} (z_1; \psi_1(x_1 x_2) u_1, w_1, w_2; \lambda) \\ &= f_1^{(b_{1,z} \cdot b_{1,z}^{-1} b_{1,0} b_{1,z})^{-1} (b_{1,0} b_{1,z})^{p+1}} (z_1; u_1, w_1, w_2; \lambda) \\ &= f_1^{(b_{1,0} b_{1,z})^p} (z_1; u_1, w_1, w_2; \lambda) \\ &= \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o \right)^p (u, z_1)(\lambda)(w_1 \otimes w_2). \end{aligned}$$

■

For a  $P(z)$ -intertwining map  $I$  of type  $\left( \begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix} \right)$  and an element  $w'_3$  of  $W'_3$ , the element  $\lambda_{I, w'_3} \in \text{COM}$  also have the following property:

**Proposition 7.0.17.** *Consider the subspace  $W_{\lambda_{I, w'_3}}$  of COM obtained by applying the coefficients of the vertex operators  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)$  for all  $u \in V$  to  $\lambda_{I, w'_3}$ . Then  $W_{\lambda_{I, w'_3}}$  equipped with  $Y_{P(z)}^{(g_1 g_2)^{-1}}$  is a generalized  $(g_1 g_2)^{-1}$ -twisted  $V$ -module.*

*Proof.* The proof of this result is a straightforward verification. Proposition 6.0.6 and the fact that on  $W_{\lambda_{I, w'_3}}$ ,  $Y_{P(z)}^{(g_1 g_2)^{-1}} = Y_{W_1 \boxtimes_{P(z)} W_2}^{(g_1 g_2)^{-1}}$  would help. ■

For any  $\lambda \in \text{COM}$ , let  $W_\lambda$  be the smallest subspace of  $(W_1 \otimes W_2)^*$  which contains  $\lambda$  and is stable under the action of all the coefficients of the vertex operators  $Y_{P(z)}^{(g_1 g_2)^{-1}}(u, x)$  for all  $u \in V$ .

We introduce the following conditions for  $\lambda \in \text{COM}$ . We say that  $\lambda$  satisfies the  **$P(z)$ -grading condition**, if  $\lambda$  is a (finite) sum of generalized eigenvectors for the operator  $L(0)'_{P(z)}$ .

By Corollary 7.0.12, for any  $\lambda$  satisfying the  $P(z)$ -grading condition, we know that  $W_\lambda$  can be decomposed into generalized eigenspaces of  $L(0)'_{P(z)}$ , which we denote as  $W_\lambda = \coprod_{n \in \mathbb{C}} (W_\lambda)_{[n]}$ . For any  $n \in \mathbb{C}$ , let  $\pi_n : \overline{W'_\lambda} \rightarrow (W'_\lambda)_{[n]}$  (or  $\pi_n : \overline{W_\lambda} \rightarrow (W_\lambda)_{[n]}$ ) be the projection map.

Let  $\mathcal{M}$  be a category of twisted  $V$ -modules.

**$P(z)$ - $\mathcal{M}$ -embeddability condition** An element  $\lambda \in \text{COM}_{P(z)}(W_1, W_2)$  is said to be  $P(z)$ - $\mathcal{M}$ -embeddable with respect to  $W_1$  and  $W_2$ , if

- There exists a subspace  $W \subset \text{COM}_{P(z)}(W_1, W_2)$ , such that  $(W, Y_{P(z)}^{(g_1 g_2)^{-1}})$  is a twisted  $V$ -module in  $\mathcal{M}$ , and  $W_\lambda \subset W$ .

We denote the subspace of  $P(z)$ - $\mathcal{M}$ -embeddable functionals in  $(W_1 \otimes W_2)^*$  as  $\text{EMB}_{P(z), \mathcal{M}}(W_1, W_2)$ , or  $\text{EMB}_{\mathcal{M}}$  for short.

**Remark 7.0.18.** Notice that if  $\lambda \in \text{EMB}_{\mathcal{M}}$  for some  $\mathcal{M}$ , then automatically  $\lambda$  satisfies the  $P(z)$ -grading condition. This is because  $\lambda \in W_\lambda \subset W$  for some twisted module  $W$  in  $\mathcal{M}$ . By Definition 2.0.1, any twisted module, in particular,  $W$ , can be decomposed into a direct sum of  $L(0)$ -generalized eigenspaces. In other words,  $\lambda \in W_\lambda \subset \bigoplus_{n \in \mathbb{C}} W_{[n]}$ .

**$P(z)$ -local grading restriction condition** An element  $\lambda \in \text{COM}_{P(z)}(W_1, W_2)$  is said to be  $P(z)$ -locally grading restricted with respect to  $W_1$  and  $W_2$ , if

1.  $\lambda$  satisfies the  $P(z)$ -grading condition.
2. For any  $n \in \mathbb{C}$ , we have

$$\dim(W_\lambda)_{[n]} < \infty \tag{7.69}$$

$$(W_\lambda)_{[n-l]} = 0, \text{ for sufficient large } l \in \mathbb{R}_+. \tag{7.70}$$

We denote the subspace of  $P(z)$ -locally grading restricted functionals in  $(W_1 \otimes W_2)^*$  as  $\text{LGR}_{P(z)}(W_1, W_2)$ , or  $\text{LGR}$  for short.

**Remark 7.0.19.** We can see that  $P(z)$ - $\mathcal{GM}_{gr}$ -embeddability condition is stronger than  $P(z)$ -local grading restriction condition. However, Theorem 7.0.21 says that they are equivalent under the condition  $\lambda \in \text{COM}_g$ , for any  $g \in \text{Aut}(V)$ . In other words,

$$\text{COM}_g \cap \text{LGR} = \text{COM}_g \cap \text{EMB}_{\mathcal{GM}_{gr}} = \text{COM}_g \cap \text{EMB}_{(\mathcal{GM}_{gr})_g}.$$

For  $\lambda \in \text{COM}$  satisfying the  $P(z)$ -grading condition, consider the map  $i : W_1 \otimes W_2 \rightarrow \overline{W'_\lambda} (= (W_\lambda)^*)$  defined by  $\langle i(w_1 \otimes w_2), \tilde{\lambda} \rangle = \tilde{\lambda}(w_1 \otimes w_2)$ , for  $w_1 \in W_1, w_2 \in W_2, \tilde{\lambda} \in W_\lambda$ .

**Lemma 7.0.20.** For any  $\lambda \in \text{COM}$  satisfying the  $P(z)$ -grading condition, and  $n \in \mathbb{C}$ , the space  $(W'_\lambda)_{[n]}$  is spanned by  $\pi_n \circ i(w_1 \otimes w_2)$ ,  $w_1 \in W_1, w_2 \in W_2$ .

*Proof.* Since  $(W'_\lambda)_{[n]} = ((W_\lambda)_{[n]})^*$ , to prove this lemma, we only need to prove that for any  $\tilde{\lambda} \in (W_\lambda)_{[n]}$ , if  $\langle \tilde{\lambda}, \pi_n \circ i(w_1 \otimes w_2) \rangle = 0$  for all  $w_1 \otimes w_2 \in W_1 \otimes W_2$ , then  $\tilde{\lambda} = 0$ . This is clear because by the definition of  $\pi_n$  and  $i$ , we know that  $\langle \tilde{\lambda}, \pi_n \circ i(w_1 \otimes w_2) \rangle = \langle \pi_n(\tilde{\lambda}), i(w_1 \otimes w_2) \rangle = \langle \tilde{\lambda}, i(w_1 \otimes w_2) \rangle = \tilde{\lambda}(w_1 \otimes w_2)$ .  $\square$

**Theorem 7.0.21.** Let  $g \in \text{Aut}(V)$ . Suppose  $\lambda \in \text{COM}_g$  satisfies the  $P(z)$ -grading condition. Then, the space  $W_\lambda$  equipped with  $Y_{P(z)}^{(g_1 g_2)^{-1}}$  is a  $g$ -twisted generalized  $V$ -module.

As a result, for any  $\lambda \in \text{COM}_g \cap \text{LGR}$ , the space  $W_\lambda$  equipped with  $Y_{P(z)}^{(g_1 g_2)^{-1}}$  is a grading restricted  $g$ -twisted generalized  $V$ -module. In particular, for any  $\lambda \in \text{COMP} \cap \text{LGR}$ , the space  $W_\lambda$  equipped with  $Y_{P(z)}^{(g_1 g_2)^{-1}}$  is a grading restricted  $(g_1 g_2)^{-1}$ -twisted generalized  $V$ -module.

*Proof.* It is sufficient to prove this statement for only those generalized eigenvector  $\lambda$  of  $L(0)'_{P(z)}$ . Let  $\tilde{\lambda} \in W_\lambda$ .

The equivariant property follows from the definition (7.46) of  $\text{COM}_g$ . The identity property  $Y_{P(z)}^{(g_1 g_2)^{-1}}(\mathbf{1}, x)(\tilde{\lambda}) = \tilde{\lambda}$  is equivalent to  $\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(\mathbf{1}, x)(\tilde{\lambda}) = \tilde{\lambda}$ , which

follows from the definition of  $Y_{P(z)}^{(g_1 g_2)^{-1}}$ . By using Corollary 7.0.12, the  $L(0)$ -grading condition for  $Y_{P(z)}^{(g_1 g_2)^{-1}}$  on  $W_\lambda$  can be derived from the  $L(0)$ -grading condition for  $\lambda$ . The  $L(-1)$ -derivative property for  $Y_{P(z)}^{(g_1 g_2)^{-1}}$  is proven in Proposition 7.0.8.

Therefore, we only need to prove the duality property. Let  $\lambda' \in W'_\lambda$  and  $\tilde{\lambda} \in W_\lambda$  be both homogeneous, and  $u_1, u_2 \in V$  be bi-homogeneous. (In other words,  $u_1, u_2$  are generalized eigenvectors for both  $L(0)_V$  and  $g_1 g_2$ .) By Lemma 7.0.20, we can further assume  $\lambda' = \pi_{wt(\lambda')} \circ i(w_1 \otimes w_2)$ , since by linearity, the duality holding for a spanning subset of  $W'_\lambda$  implies the full duality.

Throughout the remainder of this proof, for  $z \in \mathbb{C}^\times$  and  $a \in \mathbb{C}$ , we use the notation  $z^a = e^{a \cdot l_0(z)}$  and  $\log z = l_0(z)$ .

We consider

$$\begin{aligned}
& \left\langle \lambda', \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1)(\tilde{\lambda}) \right\rangle \\
&= \sum_{a_1 \in \mathbb{C}} \sum_{j_1, j_2=0}^K \left\langle i(w_1 \otimes w_2), \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{-\Delta+a_1-1, j_2}(u_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{-a_1-1, j_1}(u_1)(\tilde{\lambda}) \right\rangle \cdot \\
& \qquad \qquad \qquad z_1^{a_1} \log(z_1)^{j_1} z_2^{\Delta-a_1} \log(z_2)^{j_2} \\
&= \sum_{a_1 \in \mathbb{C}} \sum_{j_1, j_2=0}^K \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{-\Delta+a_1-1, j_2}(u_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{-a_1-1, j_1}(u_1)(\tilde{\lambda})(w_1 \otimes w_2) \right) \cdot \\
& \qquad \qquad \qquad z_1^{a_1} \log(z_1)^{j_1} z_2^{\Delta-a_1} \log(z_2)^{j_2}
\end{aligned} \tag{7.71}$$

where  $\Delta = \text{wt}(\lambda') + \text{wt}(u_1) + \text{wt}(u_2) - \text{wt}(\tilde{\lambda}) - 2$ . By Lemma 7.0.6, we know that on the region  $|z_1| > |z_2| > |z|$ ,

$$\left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1)(\tilde{\lambda})(w_1 \otimes w_2) \tag{7.72}$$

$$\begin{aligned}
&= \sum_{a_1, a_2 \in \mathbb{C}} \sum_{j_1, j_2=0}^K \left( \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{-a_2-1, j_2}(u_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o_{-a_1-1, j_1}(u_1)(\tilde{\lambda})(w_1 \otimes w_2) \right) \cdot \\
& \qquad \qquad \qquad z_1^{a_1} \log(z_1)^{j_1} z_2^{a_2} \log(z_2)^{j_2} \tag{7.73}
\end{aligned}$$

is absolutely convergent to  $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \tilde{\lambda})$ . Notice that the right-hand side of (7.71) is a sub-series of the series (7.72). Therefore, (7.71) is also absolutely convergent on the region given by  $|z_1| > |z_2| > |z|$ . Since the summation over  $a_1 \in \mathbb{C}$  in the right-hand side of (7.71) actually can be replaced by  $a_1 \in D$  where  $D$  is a finite union of sets of the form  $r - \mathbb{N}$ ,  $r \in \mathbb{C}$ , we know that the region of absolute convergence of (7.71) can be enlarged to the region given by  $|z_1| > |z_2| > 0$ . By Lemma 7.0.6, on the region given by  $|z_i| > |z|$ ,  $|\arg(z_i - z) - \arg(z_i)| < \frac{\pi}{2}$ ,  $i = 1, 2$ ,

$$\begin{aligned} & (z_1 - z_2)^{M_{u_1, u_2}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) (\tilde{\lambda})(w_1 \otimes w_2) \\ &= (z_1 - z_2)^{M_{u_1, u_2}} f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \tilde{\lambda}) \\ &= (z_1 - z_2)^{M_{u_1, u_2}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) (\tilde{\lambda})(w_1 \otimes w_2) \end{aligned} \quad (7.74)$$

By Lemma 7.0.6, there exists  $\alpha_{a_1, j_1; a_2, j_2} \in \mathbb{C}$  such that on the region  $|z_i| > |z|$ ,  $|\arg(z_i - z) - \arg(z_i)| < \frac{\pi}{2}$ ,  $i = 1, 2$ ,

$$(z_1 - z_2)^{M_{u_1, u_2}} f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \tilde{\lambda}) = \sum_{a_1, a_2 \in D} \sum_{j_1, j_2=0}^K \alpha_{a_1, j_1; a_2, j_2} z_1^{a_1} \log(z_1)^{j_1} z_2^{a_2} \log(z_2)^{j_2},$$

where  $D$  is a finite union of sets of the form  $r - \mathbb{N}$ ,  $r \in \mathbb{C}$ . Therefore, by (7.71), (7.72), and (7.74), on the region given by  $|z_i| > |z|$ ,  $|\arg(z_i - z) - \arg(z_i)| < \frac{\pi}{2}$ ,  $i = 1, 2$ , we have

$$\begin{aligned} & (z_1 - z_2)^{M_{u_1, u_2}} \left\langle \lambda', \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) (\tilde{\lambda}) \right\rangle \\ &= \sum_{\substack{a_1, a_2 \in D \\ a_1 + a_2 = \Delta + M_{u_1, u_2}}} \sum_{j_1, j_2=0}^K \alpha_{a_1, j_1; a_2, j_2} z_1^{a_1} \log(z_1)^{j_1} z_2^{a_2} \log(z_2)^{j_2} \\ &= (z_1 - z_2)^{M_{u_1, u_2}} \left\langle \lambda', \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_1, z_1) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o(u_2, z_2) (\tilde{\lambda}) \right\rangle \end{aligned} \quad (7.75)$$

Notice that in the intermediate step of (7.75), the set

$$D' := \{(a_1, a_2) \in D \times D \mid a_1 + a_2 = M_{u_1, u_2} + \Delta\}$$

is a finite set. Define

$$f(z_1, z_2) = (z_1 - z_2)^{-M_{u_1, u_2}} \sum_{(a_1, a_2) \in D'} \sum_{j_1, j_2=0}^K \alpha_{a_1, j_1; a_2, j_2} z_1^{a_1} \log(z_1)^{j_1} z_2^{a_2} \log(z_2)^{j_2}.$$

By (7.75), we know that  $\left\langle \lambda', \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (u_2, z_2) \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (u_1, z_1) (\tilde{\lambda}) \right\rangle$  is absolutely convergent to  $f(z_1, z_2)$  on the region given by  $|z_1| > |z_2| > |z|$ ,  $|\arg(z_i - z) - \arg(z_i)| < \frac{\pi}{2}$ ,  $i = 1, 2$ , which can be enlarged to the region given by  $|z_1| > |z_2| > 0$ . Similarly, we know that  $\left\langle \lambda', \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (u_1, z_1) \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (u_2, z_2) (\tilde{\lambda}) \right\rangle$  is absolutely convergent to  $f(z_1, z_2)$  on the region given by  $|z_1| > |z_2| > 0$ .

We still need to consider

$$\begin{aligned} & \left\langle \lambda', \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (Y(u_1, z_1 - z_2) u_2, z_2) (\tilde{\lambda}) \right\rangle \\ &= \sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \geq -M_{u_1, u_2}}} \sum_{j_2=0}^K \left( \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o_{-\Delta+a_1-1, j_2} \left( (u_1)_{-a_1-1} u_2 \right) (\tilde{\lambda}) (w_1 \otimes w_2) \right) \cdot \\ & \qquad \qquad \qquad (z_1 - z_2)^{a_1} z_2^{\Delta-a_1} \log(z_2)^{j_2}, \end{aligned} \tag{7.76}$$

where  $M_{u_1, u_2} \in \mathbb{N}$ . By Proposition 7.0.7, we know that

$$\left( \left(Y_{P(z)}^{(g_1 g_2)^{-1}}\right)^o (Y(u_1, z_1 - z_2) u_2, z_2) \lambda \right) (w_1 \otimes w_2) = f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \tilde{\lambda}) \tag{7.77}$$

on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ , and  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ ,  $|\arg(z_2 - z) - \arg(z_2)| < \frac{\pi}{2}$ . Since  $(z_1 - z_2, z_2) = (0, \infty)$  is a regular singularity of  $f_2(z_1, z_2; u_1, u_2, w_1, w_2; \tilde{\lambda})$ , there exist a set  $D_1$  which is a finite union of sets of form

$r - \mathbb{N}$ ,  $r \in \mathbb{C}$ , and  $\beta_{a_1; a_2, j_2} \in \mathbb{C}$ , such that on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ , the left-hand side of (7.77) is equal to the following absolutely convergent series,

$$\sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \geq -M_{u_1, u_2}}} \sum_{a_2 \in D_1} \sum_{j_2=0}^K \beta_{a_1; a_2, j_2} (z_1 - z_2)^{a_1} z_2^{a_2} \log(z_2)^{j_2}. \quad (7.78)$$

So, we know that (7.76) is equal to

$$\sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \geq -M_{u_1, u_2}}} \sum_{j_2=0}^K \beta_{a_1; \Delta - a_1, j_2} (z_1 - z_2)^{a_1} z_2^{\Delta - a_1} \log(z_2)^{j_2}. \quad (7.79)$$

As a sub-series of (7.78), the series (7.79) is also absolutely convergent on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ . By the form of (7.79), we can see that the region where (7.79) is absolutely convergent can be enlarged to the region given by  $|z_2| > |z_1 - z_2| > 0$ . Furthermore, on the region given by  $0 < |z_1 - z_2| + |z| < |z_2|$ ,  $|z_1| > |z|$ ,  $|\arg(z_i - z) - \arg(z_i)| < \frac{\pi}{2}$ ,  $i = 1, 2$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ , we have

$$\begin{aligned} & \sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \geq -M_{u_1, u_2}}} \sum_{a_2 \in D_1} \sum_{j_2=0}^K \beta_{a_1; a_2, j_2} (z_1 - z_2)^{a_1 + M_{u_1, u_2}} z_2^{a_2} \log(z_2)^{j_2} \\ &= (z_1 - z_2)^{M_{u_1, u_2}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (Y(u_1, z_1 - z_2) u_2, z_2) (\tilde{\lambda})(w_1 \otimes w_2) \\ &= (z_1 - z_2)^{M_{u_1, u_2}} f_2^e(z_1, z_2; u_1, u_2, w_1, w_2; \tilde{\lambda}) \\ &= (z_1 - z_2)^{M_{u_1, u_2}} \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_2, z_2) \left( Y_{P(z)}^{(g_1 g_2)^{-1}} \right)^o (u_1, z_1) (\tilde{\lambda})(w_1 \otimes w_2) \\ &= \sum_{a_1, a_2 \in D} \sum_{j_1, j_2=0}^K \alpha_{a_1, j_1; a_2, j_2} z_1^{a_1} \log(z_1)^{j_1} z_2^{a_2} \log(z_2)^{j_2}. \end{aligned} \quad (7.80)$$

We further know that on the above region,

$$\sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \geq -M_{u_1, u_2}}} \sum_{j_2=0}^K \beta_{a_1; \Delta - a_1, j_2} (z_1 - z_2)^{a_1 + M_{u_1, u_2}} z_2^{\Delta - a_1} \log(z_2)^{j_2}$$

$$\begin{aligned}
&= \sum_{\substack{a_1, a_2 \in D \\ a_1 + a_2 = \Delta + M_{u_1, u_2}}} \sum_{j_1, j_2 = 0}^K \alpha_{a_1, j_1; a_2, j_2} z_1^{a_1} \log(z_1)^{j_1} z_2^{a_2} \log(z_2)^{j_2} \\
&= (z_1 - z_2)^{M_{u_1, u_2}} f(z_1, z_2).
\end{aligned} \tag{7.81}$$

So, on the above region, and therefore the enlarged region given by  $|z_2| > |z_1 - z_2| > 0$ ,  $|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}$ , we know that the left-hand side of (7.76) is equal to  $f(z_1, z_2)$ , which finishes the proof of the duality.  $\blacksquare$

**Definition 7.0.22.** *Let  $\mathcal{C}$  is a category of twisted  $V$ -modules. We say  $\mathcal{C}$  is closed under taking image if for any twisted modules  $M_1, M_2$ , and a surjective homomorphism  $M_1 \rightarrow M_2$ ,  $M_1 \in \text{ob } \mathcal{C}$  implies  $M_2 \in \text{ob } \mathcal{C}$ .*

For example, the category  $\mathcal{GM}_{gr}(G)$  is closed under taking image for any subgroup  $G \leq \text{Aut}(V)$ .

**Theorem 7.0.23.** *Suppose that  $\mathcal{C}$  is closed under taking image. Let  $W_i$  be  $g_i$ -twisted modules,  $i = 1, 2$ . Then, an element  $\lambda \in (W_1 \otimes W_2)^*$  is in  $W_1 \boxtimes_{P(z)} W_2$  if and only if  $\lambda$  satisfies the  $P(z)$ -compatibility condition and the  $P(z)$ - $\mathcal{C}$ -embeddability condition. In other words,*

$$W_1 \boxtimes_{P(z)} W_2 = \text{COMP}_{P(z)}(W_1, W_2) \cap \text{EMB}_{P(z), \mathcal{C}}(W_1, W_2).$$

*Proof.* For “only if” part, let  $\lambda \in W_1 \boxtimes_{P(z)} W_2$ . By Proposition 7.0.1, we know that  $\lambda \in \text{COMP}$ . Write  $\lambda = \lambda_{I, w'_3}$ , for some  $W_3 \in \text{ob } \mathcal{C}$ ,  $w'_3 \in W'_3$ , and twisted intertwining map  $I$  of type  $\binom{W_3}{W_1 W_2}$  (see Remark 6.0.5). Consider the linear map  $\varphi : W'_3 \rightarrow W_1 \boxtimes_{P(z)} W_2$  defined by  $\varphi(w') = \lambda_{I, w'}$ ,  $w' \in W'_3$ . By definition (6.2), Proposition 6.0.6 and 7.0.17, we know that  $\varphi$  is a module map. Since  $W_3 \in \text{ob } \mathcal{C}$ ,  $\mathcal{C}$  is closed under taking contragredient and image, we know that  $\text{image}(\varphi) \in \text{ob } \mathcal{C}$ . Since  $W_\lambda = W_{\lambda_{I, w'_3}} \subset \text{image}(\varphi)$ , we know that  $\lambda \in \text{EMB}_{\mathcal{C}}$ . Therefore,  $\lambda \in \text{COMP} \cap \text{EMB}_{\mathcal{C}}$ .

Now we prove the “if” part. Let  $\lambda \in \text{COMP} \cap \text{EMB}_{\mathcal{C}}$ .

We shall identify  $W_{\lambda}^*$  and  $\overline{W'_{\lambda}}$ . We define a linear map  $I : W_1 \otimes W_2 \rightarrow \overline{W'_{\lambda}}$  by

$$\langle \tilde{\lambda}, I(w_1 \otimes w_2) \rangle = \tilde{\lambda}(w_1 \otimes w_2), \text{ for any } \tilde{\lambda} \in W_{\lambda}.$$

Define  $\mathcal{Y}_I(\cdot, x) : W_1 \otimes W_2 \rightarrow W'_{\lambda}\{x\}[\log(x)]$  by

$$\mathcal{Y}_I(w_1, x)w_2 = x^{L_{W'_{\lambda}}(0)} z^{-L_{W'_{\lambda}}(0)} I(x^{-L_{W_1}(0)} z^{L_{W_1}(0)} w_1 \otimes x^{-L_{W_2}(0)} z^{L_{W_2}(0)} w_2).$$

We claim that  $\mathcal{Y}_I$  is a twisted intertwining operator of type  $(\frac{W'_{\lambda}}{W_1 W_2})$ .

For  $u_1, \dots, u_{k-1} \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$  and  $w \in W_{\lambda}$ , we have

$$\begin{aligned} & \langle w, Y_{P(z)}^{(g_1 g_2)^{-1}}(u_1, z_1) \cdots Y_{P(z)}^{(g_1 g_2)^{-1}}(u_{k-1}, z_{k-1}) \mathcal{Y}_I(w_1, z_k) w_2 \rangle \\ &= \left\langle w, Y_{P(z)}^{(g_1 g_2)^{-1}}(u_1, z_1) \cdots Y_{P(z)}^{(g_1 g_2)^{-1}}(u_{k-1}, z_{k-1}) \right. \\ & \quad \left. \cdot e^{(\log z_k - \log z)L(0)} I(e^{-(\log z_k - \log z)L(0)} w_1 \otimes e^{-(\log z_k - \log z)L(0)} w_2) \right\rangle \\ &= \left\langle e^{(\log z_k - \log z)L(0)} w, Y_{P(z)}^{(g_1 g_2)^{-1}}(z_k^{-L(0)} z^{L(0)} u_1, e^{-(\log z_k - \log z - \log z_1)}) \right. \\ & \quad \left. \cdots Y_{P(z)}^{(g_1 g_2)^{-1}}(z_k^{-L(0)} z^{L(0)} u_{k-1}, e^{-(\log z_k - \log z - \log z_{k-1})}) \right. \\ & \quad \left. \cdot I(e^{-(\log z_k - \log z)L(0)} w_1 \otimes e^{-(\log z_k - \log z)L(0)} w_2) \right\rangle \\ &= \left\langle (Y_{P(z)}^{(g_1 g_2)^{-1}})^o(z_k^{-L(0)} z^{L(0)} u_{k-1}, e^{-(\log z_k - \log z - \log z_{k-1})}) \right. \\ & \quad \left. \cdots (Y_{P(z)}^{(g_1 g_2)^{-1}})^o(z_k^{-L(0)} z^{L(0)} u_1, e^{-(\log z_k - \log z - \log z_1)}) e^{(\log z_k - \log z)L(0)} w, \right. \\ & \quad \left. I(e^{-(\log z_k - \log z)L(0)} w_1 \otimes e^{-(\log z_k - \log z)L(0)} w_2) \right\rangle \\ &= \left( (Y_{P(z)}^{(g_1 g_2)^{-1}})^o(z_k^{-L(0)} z^{L(0)} u_{k-1}, e^{-(\log z_k - \log z - \log z_{k-1})}) \right. \\ & \quad \left. \cdots (Y_{P(z)}^{(g_1 g_2)^{-1}})^o(z_k^{-L(0)} z^{L(0)} u_1, e^{-(\log z_k - \log z - \log z_1)}) e^{(\log z_k - \log z)L(0)} w \right) \\ & \quad \left( e^{-(\log z_k - \log z)L(0)} w_1 \otimes e^{-(\log z_k - \log z)L(0)} w_2 \right). \end{aligned} \tag{7.82}$$

By (7.12), the right-hand side of (7.82) is equal to the series obtained by expanding

the function

$$f_{k-1}^e(\xi_1, \dots, \xi_{k-1}; z_k^{-L(0)} z^{L(0)} u_1, \dots, z_k^{-L(0)} z^{L(0)} u_{k-1}, \\ z_k^{-L(0)} e^{(\log z)L(0)} w_1, z_k^{-L(0)} e^{(\log z)L(0)} w_2; w)$$

on the region  $|\xi_1| > \dots > |\xi_{k-1}| > |z|$  as series in powers of  $\xi_1, \dots, \xi_{k-1}$  and non-negative integer powers of  $\log \xi_1, \dots, \log \xi_{k-1}$  and then substituting  $e^{-n(\log z_k - \log z - \log z_1)}, \dots, e^{-n(\log z_k - \log z - \log z_{k-1})}$  ( $n \in \mathbb{C}$ ) for  $e^{n \log \xi_1}, \dots, e^{n \log \xi_{k-1}}$  and  $\log z_k - \log z - \log z_1, \dots, \log z_k - \log z - \log z_{k-1}$  for  $\log \xi_1, \dots, \log \xi_{k-1}$ , respectively. Then the right-hand side of (7.82) is absolutely convergent on the region  $|z z_1 z_k^{-1}| > \dots > |z z_{k-1} z_k^{-1}| > |z|$  or equivalently the region  $|z_1| > \dots > |z_{k-1}| > |z_k| > 0$  and can be analytically extended to a multivalued analytic function on  $(\xi_1, \dots, \xi_{k-1}) = (z z_1 z_k^{-1}, \dots, z z_{k-1} z_k^{-1}) \in F_{k-1}(\mathbb{C} - \{0, z\})$  satisfies Condition 3.0.13 for  $(k-1, 0, \{0, z\})$ . Since  $w$  is  $P(z)$ -compatible, by the part 3. in the definition of  $P(z)$ -compatibility, this function satisfies Condition 3.0.13 with  $(k-1, 1, \{0\})$ . Therefore, we know that,

$$\langle w, Y_{P(z)}^{(g_1 g_2)^{-1}}(u_1, z_1) \cdots Y_{P(z)}^{(g_1 g_2)^{-1}}(u_{k-1}, z_{k-1}) \mathcal{Y}_I(w_1, z_k) w_2 \rangle$$

is absolutely convergent on the region  $|z_1| > \dots > |z_{k-1}| > |z_k| > 0$  and can be analytically extended to a multivalued analytic function on  $F_k(\mathbb{C} - \{0\})$  satisfies Condition 3.0.13 for  $(k-1, 1, \{0\})$ .

We now prove the duality property for  $\mathcal{Y}_I$ . By (7.82), for  $v \in V$ ,  $w_1 \in W_1$  and  $w_2 \in W_2$ ,

$$\langle w, Y_{P(z)}^{(g_1 g_2)^{-1}}(v, z_1) \mathcal{Y}_I(w_1, z_2) w_2 \rangle \\ = \left( (Y_{P(z)}^{(g_1 g_2)^{-1}})^o(z_2^{-L(0)} z^{L(0)} v, e^{-(\log z_2 - \log z - \log z_1)}) e^{(\log z_2 - \log z)L(0)} w \right) \\ \left( e^{-(\log z_2 - \log z)L(0)} w_1 \otimes e^{-(\log z_2 - \log z)L(0)} w_2 \right),$$

which is absolutely convergent on the region  $|z_2^{-1}zz_1| > |z|$  or equivalently  $|z_1| > |z_2| > 0$  and its sum is equal to

$$\begin{aligned} f_1^e(e^{-(\log z_2 - \log z - \log z_1)}, z_2^{-L(0)} z^{L(0)} v, e^{-(\log z_2 - \log z)L(0)} w_1, \\ e^{-(\log z_2 - \log z)L(0)} w_2; e^{(\log z_2 - \log z)L(0)} w) \end{aligned} \quad (7.83)$$

on the region  $|z_2^{-1}zz_1| > |z|$ ,  $|\arg(z_2^{-1}zz_1 - z) - \arg z_2^{-1}zz_1| < \frac{\pi}{2}$  or equivalently,  $|z_1| > |z_2| > 0$ ,  $|\arg(z_1 - z_2) - \arg z_1|$ . By definition,

$$\begin{aligned} \langle w, \mathcal{Y}_I(w_1, z_2) Y^{g_2}(v, z_1) w_2 \rangle \\ = \langle e^{(\log z_2 - \log z)L(0)} w, \mathcal{Y}_I(e^{-(\log z_2 - \log z)L(0)} w_1, z) \\ \cdot Y^{g_2}(z_2^{-L(0)} z^{L(0)} v, e^{-(\log z_2 - \log z - \log z_1)}) e^{-(\log z_2 - \log z)L(0)} w_2 \rangle \\ = (e^{(\log z_2 - \log z)L(0)} w) (e^{-(\log z_2 - \log z)L(0)} w_1 \\ \otimes Y^{g_2}(z_2^{-L(0)} z^{L(0)} v, e^{-(\log z_2 - \log z - \log z_1)}) e^{-(\log z_2 - \log z)L(0)} w_2) \end{aligned}$$

is absolutely convergent on the region  $|z| > |z_2^{-1}zz_1| > 0$  or equivalently  $|z_2| > |z_1| > 0$  and its sum is equal to to (7.83) on the region  $|z_2| > |z_1| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_2^{-1}zz_1 - z) - \arg z < -\frac{\pi}{2}$  or equivalently,  $|z_2| > |z_1| > 0$ ,  $-\frac{3\pi}{2} < \arg(z_1 - z_2) - \arg z_2 < -\frac{\pi}{2}$ . By 2.(a) in the compatibility condition, we see that

$$\begin{aligned} \langle w, \mathcal{Y}_I(Y^{g_1}(v, z_1 - z_2) w_1, z_2) w_2 \rangle \\ = \langle e^{(\log z_2 - \log z)L(0)} w, \mathcal{Y}_I(Y^{g_1}(z_2^{-L(0)} z^{L(0)} v, z_2^{-1}zz_1 - z) e^{-(\log z_2 - \log z)L(0)} w_1, z) \\ e^{-(\log z_2 - \log z)L(0)} w_2 \rangle \\ = (e^{(\log z_2 - \log z)L(0)} w) (Y^{g_1}(z_2^{-L(0)} z^{L(0)} v, z_2^{-1}zz_1 - z) e^{-(\log z_2 - \log z)L(0)} w_1 \\ \otimes e^{-(\log z_2 - \log z)L(0)} w_2), \end{aligned}$$

which is absolutely convergent on the region  $|z| > |z_2^{-1}zz_1 - z| > 0$  or equivalently

$|z_2| > |z_1 - z_2| > 0$  and its sum is equal to (7.83) on the region  $|z| > |z_2^{-1}zz_1 - z| > 0$ ,  $|\arg z_2^{-1}zz_1 - \arg z| < \frac{\pi}{2}$  or equivalently  $|z_2| > |z_1 - z_2| > 0$ ,  $|\arg z_1 - \arg z_2| < \frac{\pi}{2}$ .

Thus the duality property for  $\mathcal{Y}_I$  is proven. Since

$$\begin{aligned}
& \frac{d}{dx} \mathcal{Y}_I(w_1, x)w_2 \\
&= \frac{d}{dx} x^{L(0)} z^{-L(0)} I(x^{-L(0)} z^{L(0)} w_1 \otimes x^{-L(0)} z^{L(0)} w_2) \\
&= x^{-1} x^{L(0)} z^{-L(0)} I(x^{-L(0)} z^{L(0)} w_1 \otimes x^{-L(0)} z^{L(0)} w_2) \\
&\quad - x^{L(0)} z^{-L(0)} I(x^{-1} x^{-L(0)} z^{L(0)} w_1 \otimes x^{-L(0)} z^{L(0)} w_2) \\
&\quad - x^{L(0)} z^{-L(0)} I(x^{-L(0)} z^{L(0)} w_1 \otimes x^{-1} x^{-L(0)} z^{L(0)} w_2) \\
&= x^{-1} (L(0) \mathcal{Y}_I(w_1, x)w_2 - \mathcal{Y}_I(w_1, x)L(0)w_2 - \mathcal{Y}_I(L(0)w_1, x)w_2), \tag{7.84}
\end{aligned}$$

the  $L(-1)$ -derivative property can be derived from the  $L(0)$ -commutator formula. Since the duality of  $\mathcal{Y}_I$  is already proven, similar as the proof of Proposition 7.0.10, a Jacobi identity for  $Y_{P(z)}^{(g_1, g_2)^{-1}}(v, z_1)$  and  $\mathcal{Y}_I(w_1, z_2)$  can be proven for  $v \in V^{(g_1, g_2)}$ . By letting  $v = \omega \in V^{\text{Aut}(V)} \subset V^{(g_1, g_2)}$ , we know that the Virasoro-bracket relation holds. In particular, the  $L(0)$ -commutator formula for  $\mathcal{Y}_I$  holds, which implies the  $L(-1)$ -derivative property for  $\mathcal{Y}_I$ . So, we have proved that  $\mathcal{Y}_I$  is a twisted intertwining operator. Then  $I$  is a twisted  $P(z)$ -intertwining map.

Since  $\lambda \in \text{EMB}_{\mathcal{C}}$ , there exists a subspace  $\tilde{W}$  of  $\text{COM}$ , such that  $(\tilde{W}, Y_{P(z)}^{(g_1, g_2)^{-1}}) \in \text{ob } \mathcal{C}$ , and  $W_\lambda \subset \tilde{W}$ . Let  $i : W_\lambda \rightarrow \tilde{W}$  be the inclusion map, and  $p : \tilde{W}' \rightarrow W'_\lambda$  be the adjoint of  $i$ . Since  $\tilde{W} \in \text{ob } \mathcal{C}$  and  $\mathcal{C}$  is closed under taking contragredient, we know that  $\tilde{W}' \in \text{ob } \mathcal{C}$ . Since  $p$  is a surjective module map,  $\tilde{W} \in \text{ob } \mathcal{C}$ , and  $\mathcal{C}$  is closed under taking image, we know that  $W'_\lambda \in \text{ob } \mathcal{C}$ . Now, we have

$$\lambda(w_1 \otimes w_2) = \langle \lambda, I(w_1 \otimes w_2) \rangle = \lambda_{I, \lambda}(w_1 \otimes w_2), \tag{7.85}$$

where  $\lambda \in W_\lambda \subset (W'_\lambda)'$ ,  $W'_\lambda \in \text{ob } \mathcal{C}$ , and  $I$  is a twisted  $P(z)$ -intertwining map of type

$(\begin{smallmatrix} W' \\ W_1 W_2 \end{smallmatrix})$ . Therefore,  $\lambda \in W_1 \boxtimes_{P(z)} W_2$ . ■

By Theorem 7.0.23, Remark 7.0.19 and Corollary 7.0.16, we have the following corollary.

**Corollary 7.0.24.** *Let  $G \leq \text{Aut}(V)$ . For the case  $\mathcal{C} = \mathcal{GM}_{gr}(G)$ , we have*

$$W_1 \boxtimes_{P(z)} W_2 = \text{COMP}_{P(z)}(W_1, W_2) \cap \text{LGR}_{P(z)}(W_1, W_2).$$

**Remark 7.0.25.** *Although for any twisted module  $W$  and  $h \in \text{Aut}(V)$ , the underlying vector spaces of  $W$  and  $\phi_h(W)$  are the same, the spaces  $\text{COMP}_{P(z)}(\phi_{h_1}(W_1), \phi_{h_2}(W_2))$  are different for different  $h_1, h_2 \in \text{Aut}(V)$ .*

Now we consider the case when  $h_1 = g_1^{k_1}$ ,  $h_2 = g_2^{k_2}$  for  $k_1, k_2 \in \mathbb{Z}$ . For  $\lambda \in (W_1 \otimes W_2)^*$ ,  $\lambda$  is  $P(z)$ -compatible with respect to  $\phi_{g_1^{k_1}}(W_1)$  and  $\phi_{g_2^{k_2}}(W_2)$  (in other words,  $\lambda \in \text{COMP}_{P(z)}(\phi_{g_1^{k_1}}(W_1), \phi_{g_2^{k_2}}(W_2))$ ) if and only if  $\lambda$  satisfies the same  $P(z)$ -compatibility condition with respect to  $W_1$  and  $W_2$ , except for (7.9) and (7.10) being replaced by

$$\begin{aligned} & f\left(\begin{smallmatrix} b_{1,z}^{k_1-1} b_{1,0}^{-k_2} b_{1,z} \\ b_{l,z}^{k_1-1} b_{l,0}^{-k_2} b_{l,z} \end{smallmatrix}\right) (z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda), \text{ and} \\ & f\left(\begin{smallmatrix} b_{1,z}^{k_1-1} b_{1,0}^{-k_2} \\ b_{l,z}^{k_1-1} b_{l,0}^{-k_2} \end{smallmatrix}\right) (z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda), \end{aligned} \quad (7.86)$$

respectively. (See Lemma 3.0.30 and Lemma 3.0.33)

## Chapter 8

# Major Assumption: Convergence and Extension Property

In this chapter, let  $W_i$  be a  $g_i$ -twisted modules in  $\mathcal{C}$ ,  $i = 1, 2, 3$ ,  $W_4$  be a  $g_1g_2g_3$ -twisted module in  $\mathcal{C}$ ,  $M_1$  be a  $g_2g_3$ -twisted module in  $\mathcal{C}$ , and  $M_2$  be a  $g_1g_2$ -twisted module in  $\mathcal{C}$ . Let  $\mathcal{Y}_i$ ,  $i = 1, 2, 3, 4$ , be twisted intertwining operators of type  $\binom{W_4}{W_1M_1}$ ,  $\binom{M_1}{W_2W_3}$ ,  $\binom{W_4}{M_2W_3}$ ,  $\binom{M_2}{W_1W_2}$ , respectively. In this section, we consider the following multiple series on  $\zeta_1, \zeta_2, z_1, \dots, z_l$ ,

$$\begin{aligned} & \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) Y^{g_3}(u_1, z_1) \cdots Y^{g_3}(u_l, z_l) w_3 \rangle \\ = & \sum_{\beta_1, \beta_2, \alpha_1, \dots, \alpha_l \in \mathbb{C}} \sum_{q_1, q_2, p_1, \dots, p_l \in \mathbb{N}} \left\langle w'_4, (\mathcal{Y}_1)_{-\beta_1-1, q_1}(w_1) (\mathcal{Y}_2)_{-\beta_2-1, q_2}(w_2) \right. \\ & \left. Y_{-\alpha_1-1, p_1}^{g_3}(u_1) \cdots Y_{-\alpha_l-1, p_l}^{g_3}(u_l) w_3 \right\rangle \cdot \\ & \zeta_1^{\beta_1} \zeta_2^{\beta_2} z_1^{\alpha_1} z_1^{\alpha_1} \cdots z_l^{\alpha_l} \cdot \log(\zeta_1)^{q_1} \log(\zeta_2)^{q_2} \log(z_1)^{p_1} \cdots \log(z_l)^{p_l}. \end{aligned}$$

For a twisted module  $W$ , we define  $P(W) = \{n \in \mathbb{C} | W_{[n]} \neq 0\}$ . For any  $g \in \text{Aut}(V)$ ,  $\mu = a + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$ , we define

$$V^{[\mu, g]} = \{v \in V \mid (g - e^{2\pi i a})^k v = 0, \text{ for some } k \in \mathbb{N}\},$$

$$P_g(V) = \{a \in \mathbb{C} \mid \Im(a) \in [0, 1) \text{ and } V^{[a+\mathbb{Z}, g]} \neq 0\}.$$

We say that  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies **the regular singular point property for products**, if for any  $a_i \in P(W_i)$ ,  $i = 1, 2, 3, 4$ , and

$$a \in \text{Span}_{\mathbb{N}} (P_{g_1}(V) \cup P_{g_2}(V) \cup P_{g_3}(V) \cup P_{g_2g_3}(V) \cup P_{g_1g_2g_3}(V)),$$

the set

$$\left\{ \mu \in \mathbb{C}/\mathbb{Z} \left| \begin{array}{l} \text{The map } (\mathcal{Y}_1)_{n_1, k_1}(\cdot)(\mathcal{Y}_2)_{n_2, k_2}(\cdot)(\cdot) \neq 0, \text{ for some } n_1 \in \mu, \\ n_2 \in (-a + a_1 + a_2 + a_3 - a_4 + \mathbb{Z}) - \mu, k_1, k_2 \in \mathbb{N}. \end{array} \right. \right\}$$

is finite. Similarly, we say that  $(\mathcal{Y}_3, \mathcal{Y}_4)$  satisfies **the regular singular point property for iterate**, if for any  $a_i \in P(W_i)$ ,  $i = 1, 2, 3, 4$ , and

$$a \in \text{Span}_{\mathbb{N}} (P_{g_1}(V) \cup P_{g_2}(V) \cup P_{g_3}(V) \cup P_{g_1g_2}(V) \cup P_{g_1g_2g_3}(V)),$$

the set

$$\left\{ \mu \in \mathbb{C}/\mathbb{Z} \left| \begin{array}{l} \text{The map } (\mathcal{Y}_3)_{n_1, k_1}((\mathcal{Y}_4)_{n_2, k_2}(\cdot)(\cdot))(\cdot) \neq 0, \text{ for some } n_1 \in \mu, \\ n_2 \in (-a + a_1 + a_2 + a_3 - a_4 + \mathbb{Z}) - \mu, k_1, k_2 \in \mathbb{N}. \end{array} \right. \right\}$$

is finite.

**Assumption 8.0.1.** *Throughout the remainder of this work, we assume that any pair of twisted intertwining operators that is considered satisfies the regular singular point property for both product and iterate.*

**Remark 8.0.2.** *A pair of a twisted intertwining operator and a vertex operator acting on module automatically satisfies the regular singular point property for both product and iterate.*

If a twisted intertwining operator  $\mathcal{Y}$  satisfies the condition that the set

$$\{\mu \in \mathbb{C}/\mathbb{Z} \mid \mathcal{Y}_{n,k} \neq 0 \text{ for some } n \in \mu, k \in \mathbb{N}\} \text{ is finite,} \quad (8.1)$$

then  $(\mathcal{Y}, \mathcal{Y}_1)$  and  $(\mathcal{Y}_1, \mathcal{Y})$  satisfies the regular singular point property for both product and iterate, for any twisted intertwining operator  $\mathcal{Y}_1$  of suitable type.

Therefore, by (3.52), we know that if sets

$$\{\mu \in \mathbb{C}/\mathbb{Z} \mid n \in \mu \text{ for some } n \in P(W_i)\}, \quad i = 1, 2, 3, \quad \text{are finite,} \quad (8.2)$$

then any twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1 W_2}$  satisfies condition (8.1). Therefore, any twisted intertwining operator pair with a twisted intertwining operator  $\mathcal{Y}$  of type  $\binom{W_3}{W_1 W_2}$  satisfies the regular singular point property for both product and iterate.

If  $g \in \text{Aut}(V)$  is of finite order, then for any indecomposable  $g$ -twisted  $V$ -module  $W$ , the set  $\{\mu \in \mathbb{C}/\mathbb{Z} \mid n \in \mu \text{ for some } n \in P(W)\}$  is finite. Notice that  $Y_W^g(v, x)w \in W[[x^{\frac{1}{o(g)}}, x^{\frac{-1}{o(g)}}]]$ , for any  $v \in V$ ,  $w \in W$ , where  $o(g)$  is the order of  $g$ . We know that for any  $g$ -twisted module  $W$ , and  $a \in \mathbb{C}$ ,  $\bigoplus_{n \in a + \frac{1}{o(g)}\mathbb{Z}} W_{[n]}$  is a submodule, and therefore a direct summand of  $W$ .

We say that  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies **the convergence and extension property for products**, if the followings are satisfied.

- (1) *convergence condition for products*: For any  $l \in \mathbb{N}$ ,  $u_1, \dots, u_l \in V$ ,  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , the series,

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) Y^{g_3}(u_1, z_1) \cdots Y^{g_3}(u_l, z_l) w_3 \rangle, \quad (8.3)$$

is absolutely convergent on the region given by  $0 < |z_l| < \cdots < |z_1| < |\zeta_2| < |\zeta_1|$ .

- (2) The series (8.3) can be extended to a multi-valued analytic function

$$f_{\mathcal{Y}_1, \mathcal{Y}_2, l+2}(z_1, \dots, z_l, \zeta_1, \zeta_2; u_1, \dots, u_l; w_1, w_2, w_3, w'_4) \quad (8.4)$$

on  $(z_1, \dots, z_l, \zeta_1, \zeta_2) \in F_{l+2}(\mathbb{C} - \{0\})$  satisfying Condition 3.0.13 for  $(l, 2, \{0\})$ , with a preferred single valued branch  $f_{\mathcal{Y}_1, \mathcal{Y}_2, l+2}^e$  on  $(z_1, \dots, z_l, \zeta_1, \zeta_2) \in F_{l,2}^0(\mathbb{C} - \{0\})$ . Moreover, (8.3) converges to the preferred branch  $f_{\mathcal{Y}_1, \mathcal{Y}_2, l+2}^e$  on

the connected component of

$$\left\{ (z_1, \dots, z_l, \zeta_1, \zeta_2) \in F_{l,2}^0(\mathbb{C} - \{0\}) \mid 0 < |z_l| < \dots < |z_1| < |\zeta_2| < |\zeta_1| \right\}$$

containing  $(-l, -l+1, \dots, -1, -2l-3, -2l-1)$ . (8.5)

**Remark 8.0.3.** *The complex manifold with boundary,  $F_{1,2}^0(\mathbb{C} - \{0\})$ , is a simply connected region given by cutting  $(z_1, \zeta_1, \zeta_2) \in F_{1,2}(\mathbb{C} - \{0\}) (= F_3(\mathbb{C} - \{0\}))$  along  $z_1 \in \zeta_1 + \mathbb{R}_+$ ,  $z_1 \in \zeta_2 + \mathbb{R}_+$ ,  $\zeta_1 \in \zeta_2 + \mathbb{R}_+$ ,  $z_1, \zeta_1, \zeta_2 \in \mathbb{R}_+$ . By giving these cuttings on  $F_{1,2}(\mathbb{C} - \{0\})$  and getting a simply connected region, we have a unique way to identify  $\pi_1(F_{1,2}(\mathbb{C} - \{0\}), p)$  for different point  $p \in F_{1,2}(\mathbb{C} - \{0\})$ . Therefore, any loop in  $F_{1,2}(\mathbb{C} - \{0\})$  represents a group element in  $\pi_1(F_{1,2}(\mathbb{C} - \{0\})) \cong PB_4$ .*

In the definition of convergence and extension property, only the action of  $V$  on  $W_3$  is involved (in (8.3)). We also want to study the following series:

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(Y(v, z_1 - \zeta_2)w_2, \zeta_2)w_3 \rangle, \quad (8.6)$$

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) Y(v, z_1) \mathcal{Y}_2(w_2, \zeta_2)w_3 \rangle, \quad (8.7)$$

$$\langle w'_4, \mathcal{Y}_1(Y(v, z_1 - \zeta_1)w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2)w_3 \rangle, \quad (8.8)$$

$$\langle w'_4, Y(v, z_1) \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2)w_3 \rangle. \quad (8.9)$$

**Proposition 8.0.4.** *Suppose  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies the convergence and extension condi-*

tion for product, then,

- (1) The series (8.6) is absolutely convergent on the region given by  $0 < |z_1 - \zeta_2| + |\zeta_2| < |\zeta_1|$ , and  $0 < |z_1 - \zeta_2| < |\zeta_2|$ . Moreover, it is equal to  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v; w_1, w_2, w_3, w'_4)$  on the connected component of

$$\left\{ (z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\}) \mid 0 < |z_1 - \zeta_2| < |\zeta_2|, |z_1 - \zeta_2| + |\zeta_2| < |\zeta_1| \right\}$$

containing  $(-2, -5, -3)$  (or equivalently, containing  $(-4, -5, -3)$ ). (8.10)

- (2) The series (8.7) is absolutely convergent on the region given by  $0 < |\zeta_2| < |z_1| < |\zeta_1|$ . Moreover, it is equal to  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v; w_1, w_2, w_3, w'_4)$  on the connected component of

$$\left\{ (z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\}) \mid 0 < |\zeta_2| < |z_1| < |\zeta_1| \right\}$$

containing  $(-4, -5, -3)$ . (8.11)

- (3) The series (8.8) is absolutely convergent on the region given by  $0 < |\zeta_2| + |z_1 - \zeta_1| < |\zeta_1|$ . Moreover, it is equal to  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v; w_1, w_2, w_3, w'_4)$  on the connected component of

$$\left\{ (z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\}) \mid 0 < |\zeta_2| + |z_1 - \zeta_1| < |\zeta_1| \right\}$$

containing  $(-4, -5, -3)$  (or equivalently, containing  $(-6, -5, -3)$ ). (8.12)

- (4) The series (8.9) is absolutely convergent on the region given by  $0 < |\zeta_2| < |\zeta_1| < |z_1|$ . Moreover, it is equal to  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v; w_1, w_2, w_3, w'_4)$  on the connected component of

$$\left\{ (z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\}) \mid 0 < |\zeta_2| < |\zeta_1| < |z_1| \right\} \text{ containing } (-6, -5, -3).$$

*Proof.* By the duality of  $\mathcal{Y}_2$ , on the region given by  $|\zeta_2| > |z_1 - \zeta_2| > 0$ ,  $|\zeta_2| > |z_1| > 0$ , and  $-\frac{3\pi}{2} < \arg(z_1 - \zeta_2) - \arg \zeta_2 < -\frac{\pi}{2}$ ,  $|\arg z_1 - \arg \zeta_2| < \frac{\pi}{2}$  (which contains  $(z_1, \zeta_2) = (-2, -3)$ ), for any  $m'_1 \in M'_1$ , we have

$$\begin{aligned} & \langle m'_1, \mathcal{Y}_2(Y(v, z_1 - \zeta_2)w_2, \zeta_2)w_3 \rangle = \langle m'_1, \mathcal{Y}_2(w_2, \zeta_2)Y(v, z_1)w_3 \rangle \\ & = f_{\mathcal{Y}_{2,2}}^e(z_1, \zeta_2; v, w_2, w_3, m'_1). \end{aligned} \quad (8.13)$$

Notice that on the region given by  $(z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\})$ ,  $0 < |z_1| < |\zeta_2| < |\zeta_1|$ , the function  $\arg(z_1 - \zeta_2) - \arg \zeta_2$  is continuous and has image  $(-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$ . For  $(z_1, \zeta_1, \zeta_2) = (-1, -5, -3)$ ,  $\arg(z_1 - \zeta_2) - \arg \zeta_2 = -\pi \in (-\frac{3\pi}{2}, -\frac{\pi}{2})$ . Therefore, we know that for any  $(z_1, \zeta_1, \zeta_2)$  belongs to the region (8.5), we have  $-\frac{3\pi}{2} < \arg(z_1 - \zeta_2) - \arg \zeta_2 < -\frac{\pi}{2}$ . Therefore, on the region (8.5), we have

$$\begin{aligned} & f_{\mathcal{Y}_{1,2,3}}^e(z_1, \zeta_1, \zeta_2; v, w_1, w_2, w_3, w'_4) \\ & = \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1)\mathcal{Y}_2(w_2, \zeta_2)Y(v, z_1)w_3 \rangle \\ & = \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} \langle (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1)w'_4, \mathcal{Y}_2(w_2, \zeta_2)Y(v, z_1)w_3 \rangle \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1} \\ & = \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} f_{\mathcal{Y}_{2,2}}^e(z_1, \zeta_2; v, w_2, w_3, (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1)w'_4) \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1}. \end{aligned} \quad (8.14)$$

For any  $(\tilde{z}_1, \tilde{\zeta}_2) \in F_2(\mathbb{C} - 0)$ ,  $(z_1, \zeta_1, \zeta_2) = (\tilde{z}_1, \infty, \tilde{\zeta}_2)$  is a regular singularity of  $f_{\mathcal{Y}_{1,2,3}}^e(z_1, \zeta_1, \zeta_2; v; w_1, w_2, w_3, w'_4)$ . Therefore, the right-hand side of (8.14) is well-defined and absolutely convergent on the larger region given by  $(z_1, \zeta_1) \in F_2(\mathbb{C} - 0)$ , and  $|\zeta_1| > \max\{|z_1|, |\zeta_2|\}$ . By the analyticity of the both sides of (8.14), we know that

$$\begin{aligned} & f_{\mathcal{Y}_{1,2,3}}^e(z_1, \zeta_1, \zeta_2; v; w_1, w_2, w_3, w'_4) = \\ & \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} f_{\mathcal{Y}_{2,2}}^e(z_1, \zeta_2; v, w_2, w_3, (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1)w'_4) \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1} \end{aligned} \quad (8.15)$$

holds on the connected component of

$$\{(z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\}) \mid |\zeta_1| > \max\{|z_1|, |\zeta_2|\}\} \text{ containing } (-1, -5, -3). \quad (8.16)$$

(or equivalently containing  $(-2, -5, -3)$  or  $(-4, -5, -3)$ .)

In particular, on the sub-region given by the connected component of the region  $(z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\})$ ,  $|\zeta_1| > 2|\zeta_2| > 2|z_1 - \zeta_2| > 0$ ,  $|\arg z_1 - \arg \zeta_2| < \frac{\pi}{2}$  containing  $(-1, -5, -3)$ , we have

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v, w_1, w_2, w_3, w'_4) \\ &= \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} f_{\mathcal{Y}_2, 2}^e(z_1, \zeta_2; v, w_2, w_3, (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1) w'_4) \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1} \\ &= \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} \langle (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1) w'_4, \mathcal{Y}_2(Y(v, z_1 - \zeta_2) w_2, \zeta_2) w_3 \rangle \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1} \\ &= \langle w'_4, \mathcal{Y}_1(w_1, x) \mathcal{Y}_2(Y(v, z_1 - \zeta_2) w_2, \zeta_2) w_3 \rangle \Big|_{x^n = e^{n l_0(\zeta_1)}, \log x = l_0(\zeta_1)} \end{aligned} \quad (8.17)$$

Since  $\left(\frac{z_1}{\zeta_2} - 1, \frac{\zeta_1}{\zeta_2}\right) = (0, \infty)$  is a regular singularity of

$$f_{\mathcal{Y}_1, \mathcal{Y}_2, 3} \left( \frac{z_1}{\zeta_2}, \frac{\zeta_1}{\zeta_2}, 1; v, w_1, w_2, w_3, w'_4 \right),$$

we know that on the enlarged region, the connected component of the region given by  $(z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\})$ ,  $|\zeta_1| > |\zeta_2| + |z_1 - \zeta_2|$ ,  $|\zeta_2| > |z_1 - \zeta_2| > 0$ , containing  $(-1, -5, -3)$  (on which  $|\arg z_1 - \arg \zeta_2| < \frac{\pi}{2}$  automatically holds), we have

$$f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v, w_1, w_2, w_3, w'_4) = \langle w'_4, \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(Y(v, z_1 - \zeta_2) w_2, \zeta_2) w_3 \rangle, \quad (8.18)$$

where the right-hand side of (8.18) is absolutely convergent on the region given by

$|\zeta_1| > |\zeta_2| + |z_1 - \zeta_2| > 0$ ,  $|\zeta_2| > |z_1 - \zeta_2| > 0$ . The same connected component contains  $(-2, -5, -3)$  and  $(-4, -5, -3)$ , since we can find the paths  $[0, 1] \mapsto (-1 - t, -5, -3)$ , and  $[0, 1] \mapsto (-3 - e^{\pi t}, -5, -3)$  contained in the region, which is from  $(-1, -5, -3)$  to  $(-2, -5, -3)$ , and from  $(-2, -5, -3)$  to  $(-4, -5, -3)$ , respectively. Thus part (1) has been proven.

Since (8.15) holds on (8.16), on the sub-region of (8.16) given by the connected component of the region  $(z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\})$ ,  $|\zeta_1| > |z_1| > |\zeta_2| > 0$ ,  $|\arg(z_1 - \zeta_2) - \arg z_1| < \frac{\pi}{2}$  containing  $(-4, -5, -3)$ , we have

$$\begin{aligned}
& f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v, w_1, w_2, w_3, w'_4) \\
&= \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} f_{\mathcal{Y}_2, 2}^e(z_1, \zeta_2; v, w_2, w_3, (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1) w'_4) \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1} \\
&= \sum_{n_1 \in \mathbb{C}} \sum_{k_1 \in \mathbb{N}} \langle (A_{\pm} \mathcal{Y}_1)_{n_1, k_1}^{\mp o}(w_1) w'_4, Y(v, z_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle \zeta_1^{-n_1-1} \log(\zeta_1)^{k_1} \\
&= \langle w'_4, \mathcal{Y}_1(w_1, x) Y(v, z_1) \mathcal{Y}_2(w_2, \zeta_2) \rangle \Big|_{x^n = e^{n l_0(\zeta_1)}, \log x = l_0(\zeta_1)} \tag{8.19}
\end{aligned}$$

Similar as (8.18), we know that on the connected component of the region  $(z_1, \zeta_1, \zeta_2) \in F_{1,2}^0(\mathbb{C} - \{0\})$ ,  $|\zeta_1| > |z_1| > |\zeta_2| > 0$ , containing  $(-4, -5, -3)$  (on which  $|\arg(z_1 - \zeta_2) - \arg z_1| < \frac{\pi}{2}$  automatically holds),

$$f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e(z_1, \zeta_1, \zeta_2; v, w_1, w_2, w_3, w'_4) = \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) Y(v, z_1) \mathcal{Y}_2(w_2, \zeta_2) \rangle,$$

where the right-hand side is absolutely convergent on the region given by  $|\zeta_1| > |z_1| > |\zeta_2| > 0$ . Thus part (2) has been proven.

Part (3) and part (4) are similar. ■

**Assumption 8.0.5.** *For any pair of intertwining operators among modules in  $\mathcal{C}$ , both the convergence and extension condition for product and iterate hold. Furthermore,  $\mathcal{C}$  is closed under taking image.*

**Remark 8.0.6.** *A notation thing:  $\mathcal{Y}(w, z) = \mathcal{Y}^0(w, z) = \mathcal{Y}(w, x)|_{x=z} = \mathcal{Y}(w, e^{l_0(z)})$ . For any  $p \in \mathbb{Z}$ ,  $\mathcal{Y}^p(w, z) = \mathcal{Y}(w, x)|_{x=e^{lp(z)}} = \mathcal{Y}(w, x)|_{x^n=e^{nl_p(z)}, \log(x)=\log(z)+2\pi ip} = \mathcal{Y}(w, e^{lp(z)}) = \mathcal{Y}^0(w, e^{lp(z)}) = \mathcal{Y}(w, e^{2\pi ipz}) = \mathcal{Y}^0(w, e^{2\pi ipz})$ . For any  $z, n \in \mathbb{C}$ ,  $z \neq 0$ ,  $\log(z) = l_0(z)$ ,  $z^n = e^{nl_0(z)}$ .*

Now, we fix  $\tilde{z}_0, \tilde{z}_1, \tilde{z}_2 \in \mathbb{C}$  with  $\tilde{z}_0 = \tilde{z}_1 - \tilde{z}_2$ , and

$$0 < |\tilde{z}_0| < |\tilde{z}_2| < |\tilde{z}_1|. \quad (8.20)$$

Let  $\tilde{U}, \tilde{U}_0, \tilde{U}_1$  be

the connected component of

$$\{(\zeta_1, \zeta_2) \in F_{0,2}^0(\mathbb{C} - \{0\}) \mid 0 < |\zeta_1 - \zeta_2| < |\zeta_2| < |\zeta_1|\} \text{ containing } (\tilde{z}_1, \tilde{z}_2), \quad (8.21)$$

the connected component of

$$\{(\zeta_1, \zeta_2) \in F_{0,2}^0(\mathbb{C} - \{0\}) \mid 0 < |\zeta_1 - \zeta_2| < |\zeta_2|\} \text{ containing } (\tilde{z}_1, \tilde{z}_2),$$

the connected component of

$$\{(\zeta_1, \zeta_2) \in F_{0,2}^0(\mathbb{C} - \{0\}) \mid 0 < |\zeta_2| < |\zeta_1|\} \text{ containing } (\tilde{z}_1, \tilde{z}_2),$$

respectively.

**Remark 8.0.7.** *There are 3 different connected components of the region given by  $(\zeta_1, \zeta_2) \in F_{0,2}^0(\mathbb{C} - \{0\})$ ,  $0 < |\zeta_1 - \zeta_2| < |\zeta_2|$ , which are characterized by the inequalities  $|\arg \zeta_1 - \arg \zeta_2| < \frac{\pi}{2}$ ,  $-2\pi < \arg \zeta_1 - \arg \zeta_2 < -\frac{3\pi}{2}$ , and  $\frac{3\pi}{2} < \arg \zeta_1 - \arg \zeta_2 < 2\pi$ .*

*There are 3 different connected components of the region given by  $(\zeta_1, \zeta_2) \in F_{0,2}^0(\mathbb{C} - \{0\})$ ,  $0 < |\zeta_2| < |\zeta_1|$ , which are characterized by the inequalities  $|\arg(\zeta_1 - \zeta_2) - \arg \zeta_1| < \frac{\pi}{2}$ ,  $-2\pi < \arg(\zeta_1 - \zeta_2) - \arg \zeta_1 < -\frac{3\pi}{2}$ , and  $\frac{3\pi}{2} < \arg(\zeta_1 - \zeta_2) - \arg \zeta_1 < 2\pi$ .*

*There are 5 different connected components of the region given by  $(\zeta_1, \zeta_2) \in F_{0,2}^0(\mathbb{C} - \{0\})$ ,  $0 < |\zeta_1 - \zeta_2| < |\zeta_2| < |\zeta_1|$ , which are characterized by a pair of inequalities.*

In Table 8.0.7, we give names  $U_i$ ,  $i = 1, \dots, 5$ , to these connected components, and write down some points belongs to each of them, where  $\epsilon$  and  $\theta$  are any real numbers satisfying  $\epsilon \in (0, 0.5)$  and  $\theta \in (\frac{\pi}{12}, \frac{\pi}{4})$ . Therefore,  $\tilde{U} \in \{U_1, \dots, U_5\}$  is the one which contains  $(\tilde{z}_1, \tilde{z}_2)$ .

Table 8.1: Five connected components

$\arg(\zeta_1 - \zeta_2) - \arg(\zeta_1)$ $\arg(\zeta_1) - \arg(\zeta_2)$	$(-2\pi, -\frac{3\pi}{2})$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{3\pi}{2})$
$(-2\pi, -\frac{3\pi}{2})$	$\emptyset$	$U_2, (3 + \epsilon, 3e^{-\theta i})$	$\emptyset$
$(-\frac{\pi}{2}, \frac{\pi}{2})$	$U_3, (3 - \epsilon i, 3e^{-\theta i})$	$U_1, (-5, -3)$	$U_4, (3 + \epsilon, 3e^{\theta i})$
$(\frac{\pi}{2}, \frac{3\pi}{2})$	$\emptyset$	$U_5, (3 - \epsilon i, 3e^{\theta i})$	$\emptyset$

## Chapter 9

### The $P(z)$ Part

Let  $\zeta_1, \zeta_2$  be complex variables, and  $\zeta_0 = \zeta_1 - \zeta_2$ .

Since  $(\zeta_1 - \zeta_2, \zeta_2) = (0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 2}(\zeta_1, \zeta_2; w_1, w_2, w_3, w'_4)$ , for any  $\tilde{U}' \in \{U_1, \dots, U_5\}$ , there exist  $a_{n;i,j}(w_1, w_2, w_3, w'_4) \in (W_1 \otimes W_2 \otimes W_3 \otimes W'_4)^*$  for  $n \in \mathbb{C}$ ,  $i, j \in \mathbb{N}$  such that on the region given by  $(\zeta_1, \zeta_2) \in \tilde{U}'$ , for homogeneous  $w_1, w_2, w_3, w'_4$ , we have

$$\begin{aligned} \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle &= f_{\mathcal{Y}_1, \mathcal{Y}_2, 2}^{b(\tilde{U}')}(\zeta_1, \zeta_2; w_1, w_2, w_3, w'_4) \\ &= \sum_{n \in \mathbb{C}} \sum_{i, j \in \mathbb{N}} a_{n;i,j}(w_1, w_2, w_3, w'_4) e^{(-n-1)l_0(\zeta_2)} (l_0(\zeta_2))^i e^{(\Delta+n+1)l_0(\zeta_0)} (l_0(\zeta_0))^j, \end{aligned} \quad (9.1)$$

(See also Lemma 11.2 in [HLZ7])

where

$$\Delta = \text{wt } w'_4 - \text{wt } w_1 - \text{wt } w_2 - \text{wt } w_3,$$

and  $b(\tilde{U}')$  is an element in the fundamental group of  $F_{0,2}^0(\mathbb{C} - \{0\})$  which only depends on  $\tilde{U}'$ . We know that there exists  $M \in \mathbb{Z}_+$ ,  $s_1, \dots, s_M \in \mathbb{C}$ , such that

$$a_{n;i,j}(w_1, w_2, w_3, w'_4) = 0,$$

if  $-n - 1 \notin \cup_{k=1}^M (s_k + \mathbb{N})$ . We also know that  $a_{n;i,j}(w_1, w_2, w_3, w'_4) = 0$  for either  $i$  or  $j$  is sufficiently large. For any  $w_3 \in W_3$ ,  $w'_4 \in W_4$ , consider the linear functional  $\lambda_{w_3, w'_4} \in (W_1 \otimes W_2)^*$  defined by

$$\lambda_{w_3, w'_4}(w_1 \otimes w_2) = \langle w'_4, \mathcal{Y}_1(w_1, \tilde{z}_1) \mathcal{Y}_2(w_2, \tilde{z}_2) w_3 \rangle.$$

Therefore, we know that for homogeneous module elements,

$$\begin{aligned} & \lambda_{w_3, w'_4}(w_1 \otimes w_2) \\ &= \sum_{n \in \mathbb{C}} \sum_{i, j \in \mathbb{N}} a_{n;i,j}(w_1, w_2, w_3, w'_4) e^{(-n-1)l_0(\tilde{z}_2)} (l_0(\tilde{z}_2))^i e^{(\Delta+n+1)l_0(\tilde{z}_0)} (l_0(\tilde{z}_0))^j. \end{aligned} \quad (9.2)$$

We define  $\lambda_n^{(2)} = \lambda_n^{(2)}(w_3, w'_4) \in (W_1 \otimes W_2)^*$ ,  $n \in \mathbb{C}$ , as follows,

$$\begin{aligned} \lambda_n^{(2)}(w_1 \otimes w_2) &= \sum_{i, j \in \mathbb{N}} a_{n+wt w_3 - wt w'_4 - 1; i, j} (e^{-l_0(\tilde{z}_0)L(0)_s} w_1, e^{-l_0(\tilde{z}_0)L(0)_s} w_2, e^{-l_0(\tilde{z}_0)L(0)_s} w_3, \\ & e^{l_0(\tilde{z}_0)L(0)_s} w'_4) \cdot (\tilde{z}_0)^{n+wt w_3 - wt w'_4} \tilde{z}_2^{(-n - wt w_3 + wt w'_4)} (\log(\tilde{z}_2))^i (\log(\tilde{z}_0))^j. \end{aligned} \quad (9.3)$$

Therefore, we have  $\lambda_{w_3, w'_4}(w_1 \otimes w_2) = \sum_{n \in \mathbb{C}} \lambda_n^{(2)}(w_1 \otimes w_2)$ .

For any homogeneous  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , We define  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)$  to be an element in  $(W_1 \otimes W_2)^*$  with

$$\begin{aligned} & \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(w_1 \otimes w_2) = \\ & \sum_{i, j \in \mathbb{N}} a_{n+wt w_3 - wt w'_4 - 1; i, j} (e^{-l_0(\zeta_0)L(0)_s} w_1, e^{-l_0(\zeta_0)L(0)_s} w_2, e^{-l_0(\zeta_0)L(0)_s} w_3, e^{l_0(\zeta_0)L(0)_s} w'_4) \\ & \cdot e^{(n+wt w_3 - wt w'_4)l_0(\zeta_0)} e^{(-n - wt w_3 + wt w'_4)l_0(\zeta_2)} (l_0(\zeta_2))^i (l_0(\zeta_0))^j. \end{aligned} \quad (9.4)$$

Therefore,  $\lambda_n^{(2)}(w_3, w'_4) = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \tilde{z}_1, \tilde{z}_2)$ .

The main theorem that we want to prove is the following.

**Theorem 9.0.1.** *Suppose  $\tilde{U} = U_1$  (see (8.21) and the table in Remark 8.0.7). For*

any  $n \in \mathbb{C}$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , the linear functional  $\lambda_n^{(2)}(w_3, w'_4)$  is  $P(\tilde{z}_0)$ -compatible.

**Remark 9.0.2.** The linear functional  $\lambda_{w_3, w'_4} \in (W_1 \otimes W_2)^*$  actually satisfies most parts of the definition of  $P(\tilde{z}_0)$ -compatible, except for the fact that

$$f_n(z_1, \dots, z_n; u_1, \dots, u_n, w_1, w_2; \lambda_{w_3, w'_4})$$

is defined on  $F_n(\mathbb{C} - \{0, \tilde{z}_0, -\tilde{z}_2\})$ , instead of  $F_n(\mathbb{C} - \{0, \tilde{z}_0\})$ . In other words, there are extra singularities of  $f_n$  at  $z_i = -\tilde{z}_2$ ,  $i = 1, \dots, n$ .

*Proof.* By Lemma 8.0.4, we know that

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1) w_2, \zeta_2) w_3 \rangle \quad (9.5)$$

is absolutely convergent, on the region given by  $0 < |z_1| < |\zeta_2|$ ,  $|z_1| + |\zeta_2| < |\zeta_1|$ , to the multivalued analytic function  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(z_1 + \zeta_2, \zeta_1, \zeta_2; u_1, w_1, w_2, w_3, w_4)$ . As multivalued analytic functions, we have

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(z_1 + \zeta_2, \zeta_1, \zeta_2; u_1, w_1, w_2, w_3, w'_4) \\ &= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3} \left( \frac{\zeta_2}{\zeta_0} + \frac{z_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0} + 1, \frac{\zeta_2}{\zeta_0}; \zeta_0^{-L(0)} u_1, \zeta_0^{-L(0)} w_1, \zeta_0^{-L(0)} w_2, \zeta_0^{-L(0)} w_3, \zeta_0^{L(0)} w'_4 \right), \end{aligned} \quad (9.6)$$

whose right-hand side, as a function on  $\frac{z_1}{\zeta_0}$  and  $\frac{\zeta_2}{\zeta_0}$ , has a regular singularity  $(\frac{z_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}) = (0, \infty)$ . This is because since  $(a - c, b - c - 1, c) = (0, 0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(a, b, c)$ , we know that  $(a - c, c) = (0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(a, c + 1, c)$ . Therefore, on the region given by  $0 < |\frac{z_1}{\zeta_0}| < 1 < |\frac{\zeta_2}{\zeta_0}|$ ,  $0 < |z_1| < |\zeta_2|$ ,  $|z_1| + |\zeta_2| < |\zeta_1|$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , (9.5) can be written as an absolutely convergent series,

$$\begin{aligned} & \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1) w_2, \zeta_2) w_3 \rangle \\ &= \sum_{m, n \in \mathbb{C}} \sum_{i, j, k \in \mathbb{N}} b_{m, n; i, j, k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} z_1^{-m-1} \zeta_0^{m+n+2+\Delta_1} \end{aligned}$$

$$(\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k, \quad (9.7)$$

$$\Delta_1 = \Delta - \text{wt } u_1 = \text{wt } w'_4 - \text{wt } u_1 - \text{wt } w_1 - \text{wt } w_2 - \text{wt } w_3,$$

where  $b_{m,n;i,j,k} = 0$  unless  $-m, n$  are contained in a fixed finite union of sets of the form  $r + \mathbb{N}$ . The right-hand side of (9.7) is absolutely convergent on the region given by  $0 < |\frac{z_1}{\zeta_0}| < 1 < |\frac{\zeta_2}{\zeta_0}|$ , or equivalently,  $0 < |z_1| < |\zeta_0| < |\zeta_2|$ .

By the definition of  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)$ , on the region given by  $z_1 \in \mathbb{C}$ ,  $0 < |z_1| < |\zeta_1| - |\zeta_2|$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , we know that (9.7) is equal to its iterate sum  $\sum_{n \in \mathbb{C}} \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2)$ . In other words, on the region given by,  $0 < |z_1| < |\zeta_1| - |\zeta_2|$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , we have

$$\begin{aligned} & \sum_{n \in \mathbb{C}} \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) \\ &= \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1)w_2, \zeta_2)w_3 \rangle \\ &= \sum_{n \in \mathbb{C}} \left( \sum_{m \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} b_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \right. \\ & \quad \left. \zeta_2^{-n-1} z_1^{-m-1} \zeta_0^{m+n+2+\Delta_1} (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k \right) \quad (9.8) \end{aligned}$$

By the definition of  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)$ , we know that

$$\begin{aligned} & \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) \\ &= \sum_{m \in \mathbb{C}} \sum_{k \in \mathbb{N}} \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \left( w_1 \otimes (Y_{W_2}^{g_2})_{m,k}(u_1)w_2 \right) \cdot z_1^{-m-1} (\log(z_1))^k \\ &= \sum_{m \in \mathbb{C}} \sum_{k \in \mathbb{N}} \left( \sum_{i,j \in \mathbb{N}} a_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i,j} \left( w_1, (Y_{W_2}^{g_2})_{m,k}(u_1)w_2, w_3, w'_4 \right) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j \right) \\ & \quad \cdot (\zeta_0)^{n - \text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1 + m + 1} \zeta_2^{-n - \text{wt } w_3 + \text{wt } w'_4} z_1^{-m-1} (\log(z_1))^k. \quad (9.9) \end{aligned}$$

Comparing (9.8) and (9.9), for any  $z_1$  with  $0 < |z_1| < |\zeta_1| - |\zeta_2|$ , we have

$$\begin{aligned}
& \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) \\
&= \sum_{m \in \mathbb{C}} \sum_{i, j, k \in \mathbb{N}} b_{m, n + \text{wt } w_3 - \text{wt } w'_4 - 1; i, j, k}(u_1; w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k \\
&\quad \cdot \zeta_0^{n - \text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1 + m + 1} \tilde{z}_2^{-n - \text{wt } w_3 + \text{wt } w'_4} z_1^{-m - 1}. \\
&= \zeta_0^{n - \text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1} \zeta_2^{-n - \text{wt } w_3 + \text{wt } w'_4} \sum_{i \geq 0} (\log(\zeta_2))^i \left( \sum_{m \in \mathbb{C}} \sum_{j, k \in \mathbb{N}} \right. \\
&\quad \left. b_{m, n + \text{wt } w_3 - \text{wt } w'_4 - 1; i, j, k}(u_1; w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_0))^j (\log(z_1))^k \cdot \left( \frac{z_1}{\zeta_0} \right)^{-m - 1} \right)
\end{aligned} \tag{9.10}$$

Since the right-hand side of (9.7) is absolutely convergent on the region given by  $0 < |\frac{z_1}{\zeta_0}| < 1 < |\frac{\zeta_2}{\zeta_0}|$ , or equivalently  $0 < |z_1| < |\zeta_0| < |\zeta_2|$ , we know that for any  $n \in \mathbb{C}$ , both sides of (9.10) are absolutely convergent and are equal on the region given by  $0 < |z_1| < |\zeta_0|$ , because essentially (9.10) is a finite linear combination of power series on  $z_1/\zeta_0$ .

For any  $a \in \mathbb{C}$ ,  $\left(\frac{\zeta_2}{\zeta_0}, \frac{z_1}{\zeta_0}\right) = (\infty, a)$  is a regular singularity of (9.6), and also, (9.6) is analytic at  $\frac{z_1}{\zeta_0} = a$  for any  $a \neq 0, 1, -\frac{\zeta_2}{\zeta_0}$ . For any  $n \in \mathbb{C}$ ,  $i \geq 0$ , we consider the following sub-series of (9.10), which is convergent on  $0 < |z_1| < |\zeta_0|$ ,

$$\sum_{m \in \mathbb{C}} \sum_{j, k \in \mathbb{N}} b_{m, n + \text{wt } w_3 - \text{wt } w'_4 - 1; i, j, k}(u_1; w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_0))^j (\log(z_1))^k \cdot \left( \frac{z_1}{\zeta_0} \right)^{-m - 1}. \tag{9.11}$$

After fixing  $\zeta_1, \zeta_2$  with  $(\zeta_1, \zeta_2) \in \tilde{U}$ , as a function on  $z_1$ , (9.11) can be analytically extended to a multi-valued analytic function on the region given by  $z_1 \neq 0, \zeta_0, -\zeta_2$ . However, these multi-valued functions indexed by  $i$  are independent on  $\zeta_2$  when  $\zeta_0$  is fixed. In other words, after replacing  $\zeta_2$  by another complex number  $\zeta_{2 \text{ new}}$  such that  $(\zeta_{1 \text{ new}}, \zeta_{2 \text{ new}}) = (\zeta_0 + \zeta_{2 \text{ new}}, \zeta_{2 \text{ new}}) \in \tilde{U}$ , these multi-valued analytic functions (9.11)

indexed by  $i$  would not be changed. This fact means that  $z_1 = -\zeta_2$  is a removable singularity for these functions. Therefore, as a finite sum of (9.11) over  $i$ , on the region given by  $0 < |z_1| < |\zeta_0|$ , the series

$$\sum_{m \in \mathbb{C}} \sum_{i, j, k \in \mathbb{N}} b_{m, n+wt w_3-wt w'_4-1; i, j, k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-wt w_3+wt w'_4} z_1^{-m-1} \zeta_0^{-n-wt w_1-wt w_2-wt u_1+m+1} (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k \quad (9.12)$$

converges to a multi-valued analytic function  $h_{(n)}(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2)$  on the region given by  $\zeta_0, \zeta_2, z_1 \in \mathbb{C}$  with  $(\zeta_1, \zeta_2) \in \tilde{U}$  and  $z_1 \in F_1(\mathbb{C} - \{0, \zeta_0\})$ . We define  $h_{(n)}^e(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2)$  on the simply connected region given by  $(\zeta_1, \zeta_2) \in \tilde{U}$  and  $z_1 \in F_1^0(\mathbb{C} - \{0, \zeta_0\})$  to be the single valued branch of  $h_{(n)}$ , such that

$$(9.12) = \begin{cases} h_{(n)}^e(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) & \text{if } (\zeta_1, \zeta_2) \in \tilde{U} \text{ and } z_1 \in \Omega_{0,0,1}^{(1)}(\zeta_0), \\ h_{(n)}^{b_{1,\zeta_0}^{-1}}(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) & \text{if } (\zeta_1, \zeta_2) \in \tilde{U} \text{ and } z_1 \in \Omega_{0,0,1}^{(2)}(\zeta_0). \end{cases} \quad (9.13)$$

(see remark 7.0.3)

Therefore by (9.10), for any  $(\zeta_1, \zeta_2) \in \tilde{U}$ ,  $0 < |z_1| < |\zeta_0|$ , we have

$$\begin{aligned} \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) = \\ \begin{cases} h_{(n)}^e(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) & \text{if } (\zeta_1, \zeta_2) \in \tilde{U} \text{ and } z_1 \in \Omega_{0,0,1}^{(1)}(\zeta_0), \\ h_{(n)}^{b_{1,\zeta_0}^{-1}}(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) & \text{if } (\zeta_1, \zeta_2) \in \tilde{U} \text{ and } z_1 \in \Omega_{0,0,1}^{(2)}(\zeta_0). \end{cases} \end{aligned} \quad (9.14)$$

We also know that on the region given by  $0 < |z_1| < |\zeta_0|$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$  (larger than  $0 < |z_1| < |\zeta_1| - |\zeta_2|$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ ), the left-hand side and the right-hand side of (9.8) are absolutely convergent and equal.

Next, we should consider  $\lambda_{w_3, w'_4}(Y_{W_1}^{g_1}(u_1, z_1 - \tilde{z}_0)w_1 \otimes w_2)$  and  $\lambda_n^{(2)}(Y_{W_1}^{g_1}(u_1, z_1 -$

$\tilde{z}_0)w_1 \otimes w_2$ ). By Lemma 8.0.4, we know that

$$\langle w'_4, \mathcal{Y}_1(Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1, \zeta_1)\mathcal{Y}_2(w_2, \zeta_2)w_3 \rangle \quad (9.15)$$

is absolutely convergent, on the region given by  $0 < |z_1 - \zeta_0| + |\zeta_2| < |\zeta_1|$ , to the multivalued analytic function  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(z_1 + \zeta_2, \zeta_1, \zeta_2; u_1, w_1, w_2, w_3, w'_4)$ . With  $\zeta_1, \zeta_2$  fixed, as a function on  $z_1$ , this multivalued analytic function only has singularity at  $z_1 = 0, \zeta_0, -\zeta_2$ . Therefore, the corresponding iterate series

$$\sum_{n \in \mathbb{C}} \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2)$$

is absolutely convergent on the region given by  $0 < |z_1 - \zeta_0| < \min\{|0 - \zeta_0|, |-\zeta_2 - \zeta_0|\} = |\zeta_0|$ .

As multivalued functions, we have

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(z_1 + \zeta_2, \zeta_1, \zeta_2; u_1, w_1, w_2, w_3, w'_4) \\ = & f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}\left(\frac{z_1 - \zeta_0}{\zeta_0} + \frac{\zeta_2}{\zeta_0} + 1, \frac{\zeta_2}{\zeta_0} + 1, \frac{\zeta_2}{\zeta_0}; \zeta_0^{-L(0)}u_1, \zeta_0^{-L(0)}w_1, \zeta_0^{-L(0)}w_2, \zeta_0^{-L(0)}w_3, \zeta_0^{L(0)}w'_4\right), \end{aligned} \quad (9.16)$$

whose right-hand side, as function on  $\frac{z_1 - \zeta_0}{\zeta_0}$  and  $\frac{\zeta_2}{\zeta_0}$ , has a regular singularity  $(\frac{z_1 - \zeta_0}{\zeta_0}, \frac{\zeta_2}{\zeta_0}) = (0, \infty)$ . This is because since  $(a' - b, b - c - 1, c) = (0, 0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(a', b, c)$ , we have  $(a, b - c - 1, c) = (0, 0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(a + b, b, c)$ . Also since it is holomorphic at  $b - c - 1 = 0$ , we know that  $(a, c) = (0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(a + c + 1, c + 1, c)$ . Therefore, there exist  $c_{m, n; i, j, k}(u_1; w_1, w_2, w_3, w'_4) \in \mathbb{C}$  such that on the region given by  $0 < \left|\frac{z_1 - \zeta_0}{\zeta_0}\right| < 1 < 1 + \left|\frac{z_1 - \zeta_0}{\zeta_0}\right| < \left|\frac{\zeta_2}{\zeta_0}\right|$ ,  $0 < |z_1 - \zeta_0| + |\zeta_2| < |\zeta_1|$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , we

have

$$\begin{aligned}
& \langle w'_4, \mathcal{Y}_1 (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle \\
&= \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} (z_1 - \zeta_0)^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\
& \quad \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1 - \zeta_0))^k. \quad (9.17)
\end{aligned}$$

where  $c_{m,n;i,j,k} = 0$  unless  $-m, n$  are contained in a fixed finite union of sets of the form  $r + \mathbb{N}$ . The right-hand side of (9.17) is absolutely convergent on the region given by  $0 < \left| \frac{z_1 - \zeta_0}{\zeta_0} \right| < 1 < 1 + \left| \frac{z_1 - \zeta_0}{\zeta_0} \right| < \left| \frac{\zeta_2}{\zeta_0} \right|$ , or equivalently,  $0 < |z_1 - \zeta_0| < |\zeta_0|$ , and  $|z_1 - \zeta_0| + |\zeta_0| < |\zeta_2|$ .

Similar as above, we know that for  $(\zeta_1, \zeta_2) \in \tilde{U}$ , the series on  $z_1 - \zeta_0$ ,

$$\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2),$$

is absolutely convergent on the region given by  $(\zeta_1, \zeta_2) \in \tilde{U}$ ,  $0 < |z_1 - \zeta_0| < |\zeta_0|$ , and

$$\begin{aligned}
& \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2) \\
&= \sum_{m \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n+\text{wt } w_3 - \text{wt } w'_4 - 1; i,j,k}(u_1; w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j \\
& \quad \cdot (\log(z_1 - \zeta_0))^k \zeta_0^{n - \text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1 + m + 1} \zeta_2^{-n - \text{wt } w_3 + \text{wt } w'_4} (z_1 - \zeta_0)^{-m-1}. \quad (9.18)
\end{aligned}$$

Moreover, also similarly, as a function on  $z_1$ , (9.18) can be analytically extend to a multi-valued analytic function on the region  $z_1 \in F_1(\mathbb{C} - \{0, \zeta_0\})$ .

Also, on the region given by  $0 < |z_1 - \zeta_0| < |\zeta_0| < |\zeta_2|$ , we know the iterate series,

$$\begin{aligned}
& \sum_{n \in \mathbb{C}} \tilde{\lambda}_{(n)}^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2) \\
&= \sum_{n \in \mathbb{C}} \left( \sum_{m \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} (z_1 - \zeta_0)^{-m-1} \zeta_0^{m+n+2+\Delta_1} \right)
\end{aligned}$$

$$\cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1 - \zeta_0))^k \quad (9.19)$$

is absolutely convergent.

Next, we want to study the equalness between  $\lambda_{w_3, w'_4}(Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2)$  and  $\lambda_{w_3, w'_4}(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2)$ . Let the multivalued analytic function in the definition of twisted intertwining operator for  $\mathcal{Y}_1, \mathcal{Y}_2$  be  $f_{\mathcal{Y}_1}, f_{\mathcal{Y}_2}$ , respectively.

By Lemma 8.0.4, the multiple series

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1)Y_{M_1}^{g_2g_3}(u_1, z_1 + \zeta_2)\mathcal{Y}_2(w_2, \zeta_2)w_3 \rangle, \quad (9.20)$$

is absolutely convergent on the region given by  $0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|$ .

Also, on the region given by  $0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|$ ,  $0 < |z_1| < |\zeta_2|$ ,  $|z_1| + |\zeta_2| < |\zeta_1|$ ,  $|\arg(z_1 + \zeta_2) - \arg(\zeta_2)| < \frac{\pi}{2}$ , and  $|\arg(z_1 + \zeta_2) - \arg(z_1)| < \frac{\pi}{2}$ , we know that

$$\begin{aligned} & \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1)\mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1)w_2, \zeta_2)w_3 \rangle \\ &= \sum_{n \in \mathbb{C}} \sum_{k \geq 0} \left\langle (A_+(\mathcal{Y}_1)^{-o})_{-n-1, k}(w_1)(w'_4), \mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1)w_2, \zeta_2)w_3 \right\rangle e^{nl_0(\zeta_1)} (l_0(\zeta_1))^k \\ &= \sum_{n \in \mathbb{C}} \sum_{k \geq 0} f_{\mathcal{Y}_2, 2}^e(z_1 + \zeta_2, \zeta_2; u_1, w_2, w_3, (A_+(\mathcal{Y}_1)^{-o})_{-n-1, k}(w_1)(w'_4)) e^{nl_0(\zeta_1)} (l_0(\zeta_1))^k. \\ &= \sum_{n \in \mathbb{C}} \sum_{k \geq 0} \left\langle (A_+(\mathcal{Y}_1)^{-o})_{-n-1, k}(w_1)(w'_4), Y_{M_1}^{g_1g_2}(u_1, z_1)\mathcal{Y}_2(w_2, \zeta_2)w_3 \right\rangle e^{nl_0(\zeta_1)} (l_0(\zeta_1))^k \\ &= \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1)Y_{M_1}^{g_2g_3}(u_1, z_1 + \zeta_2)\mathcal{Y}_2(w_2, \zeta_2)w_3 \rangle. \end{aligned} \quad (9.21)$$

Let  $\phi(z_1, \zeta_1, \zeta_2) = (z_1 + \zeta_2, \zeta_1, \zeta_2)$  on  $\mathbb{C}^3$ , and  $O = \phi^{-1}(F_{1,2}^0(\mathbb{C} - \{0\}))$  with the topology such that  $\phi$  is a homeomorphism. In other words,  $O$  is the region (set-theoretically and topologically) given by

$$(z_1, \zeta_1, \zeta_2) \in \mathbb{C}^3, \quad z_1 \neq 0, \quad z_1 \neq -\zeta_2, \quad z_1 \neq \zeta_0, \quad \zeta_1 \neq 0, \quad \zeta_2 \neq 0, \quad \zeta_1 \neq \zeta_2,$$

with cuttings  $z_1 \in \mathbb{R}_+$ ,  $z_1 \in -\zeta_2 + \mathbb{R}_+$ ,  $z_1 \in \zeta_0 + \mathbb{R}_+$ ,  $\zeta_1 \in \mathbb{R}_+$ ,  $\zeta_2 \in \mathbb{R}_+$ ,  $\zeta_1 \in \zeta_2 + \mathbb{R}_+$ .

Define the region  $O_1$  by  $0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|$ ,  $0 < |z_1| < |\zeta_2|$ ,  $|z_1| + |\zeta_2| < |\zeta_1|$ ,  $|\arg(z_1 + \zeta_2) - \arg(\zeta_2)| < \frac{\pi}{2}$ ,  $|\arg(z_1 + \zeta_2) - \arg(z_1)| < \frac{\pi}{2}$ ,  $-\frac{3\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_1) < -\frac{\pi}{2}$ , and  $(\zeta_1, \zeta_2) \in \tilde{U}$ .

Define the region  $O_2$  by  $0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|$ ,  $0 < |z_1| < |\zeta_2|$ ,  $|z_1| + |\zeta_2| < |\zeta_1|$ ,  $|\arg(z_1 + \zeta_2) - \arg(\zeta_2)| < \frac{\pi}{2}$ ,  $|\arg(z_1 + \zeta_2) - \arg(z_1)| < \frac{\pi}{2}$ ,  $\frac{\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_1) < \frac{3\pi}{2}$ , and  $(\zeta_1, \zeta_2) \in \tilde{U}$ .

Notice that  $O_1$  and  $O_2$  are both connected sub-regions of  $O$ , if not empty. At least one of  $O_1$  and  $O_2$  has nonempty interior, no matter  $\tilde{U}$  is which one in  $\{U_i | i = 1, \dots, 5\}$ .

We know that  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(z_1 + \zeta_2, \zeta_1, \zeta_2; u_1, w_1, w_2, w_3, w'_4)$  has single valued branches on  $O$ . We defined  $h^e(z_1, \zeta_1, \zeta_2)$  to be the unique single valued analytic function on  $O$ ,

$$(9.21) = \begin{cases} h^e(z_1, \zeta_1, \zeta_2) & \text{if } (z_1, \zeta_1, \zeta_2) \in O_1 \\ h^{b^{-1}, \zeta_0}(z_1, \zeta_1, \zeta_2) & \text{if } (z_1, \zeta_1, \zeta_2) \in O_2 \end{cases}$$

We first assume  $\text{Int}(O_1) \neq \emptyset$ , which happens for  $\tilde{U} \in \{U_1, U_3, U_5\}$ .

Therefore, on the connected component  $O_3$  of  $\{(z_1, \zeta_1, \zeta_2) \in O | 0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|\}$  containing  $O_1$ , we know that

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) Y_{M_1}^{g_2 g_3}(u_1, z_1 + \zeta_2) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle = h^e(z_1, \zeta_1, \zeta_2). \quad (9.22)$$

Actually  $O_3$  can be characterized by inequalities as the following.

$$O_3 = \left\{ (z_1, \zeta_1, \zeta_2) \in O \left| \begin{array}{l} 0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|, (\zeta_1, \zeta_2) \in \tilde{U}_1, \\ |\arg(z_1) - \arg(z_1 + \zeta_2)| < \frac{\pi}{2}, \\ -\frac{3\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_1) < -\frac{\pi}{2}. \end{array} \right. \right\}$$

On the connected component  $O_4$  of

$$\{(z_1, \zeta_1, \zeta_2) \in O | |z_1| + |\zeta_2| < |\zeta_1|, 0 < |z_1| < |\zeta_2|\}$$

containing  $O_1$ , we know that

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1) w_2, \zeta_2) w_3 \rangle = h^e(z_1, \zeta_1, \zeta_2), \quad (9.23)$$

Therefore, by (9.7), (9.23) implies that on the region  $O_5$  given by the union of connected components of

$$\{(z_1, \zeta_1, \zeta_2) \in O \mid 0 < |z_1| < |\zeta_0| < |\zeta_2|\}$$

which have nonempty intersection with  $O_1$ , we have

$$\begin{aligned} & \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} b_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} z_1^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\ & \qquad \qquad \qquad (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k \\ & = h^e(z_1, \zeta_1, \zeta_2). \end{aligned} \quad (9.24)$$

Actually  $O_5$  can be characterized by inequalities as the following.

$$\begin{aligned} & O_5 \\ & = \left\{ \begin{array}{l} (z_1, \zeta_1, \zeta_2) \left| \begin{array}{l} 0 < |z_1| < |\zeta_0| < |\zeta_2|, |\arg(z_1 + \zeta_2) - \arg(\zeta_2)| < \frac{\pi}{2}, \\ \in O \left| -\frac{3\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_0) < -\frac{\pi}{2}, (\zeta_1, \zeta_2) \in \tilde{U}_0. \end{array} \right. \end{array} \right\}, \text{ if } \tilde{U} = U_1 \text{ or } U_5, \\ & = \left\{ \begin{array}{l} (z_1, \zeta_1, \zeta_2) \left| \begin{array}{l} 0 < |z_1| < |\zeta_0| < |\zeta_2|, |\arg(z_1 + \zeta_2) - \arg(\zeta_2)| < \frac{\pi}{2}, \\ \in O \left| \frac{\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_0) < \frac{3\pi}{2}, (\zeta_1, \zeta_2) \in \tilde{U}_0. \end{array} \right. \end{array} \right\}, \text{ if } \tilde{U} = U_3. \end{aligned}$$

Similar as (9.21), on the region  $O_6$  given by  $0 < |\zeta_2| < |z_1 + \zeta_2| < |\zeta_1|$ ,  $0 < |z_1 - \zeta_0| + |\zeta_2| < |\zeta_1|$ , and  $-\frac{3\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_1) < -\frac{\pi}{2}$ ,  $|\arg(z_1 + \zeta_2) - \arg(\zeta_1)| < \frac{\pi}{2}$ , the two sides of the following equality are absolutely convergent as multiple series,

and are equal.

$$\begin{aligned} & \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) Y_M^{g_2 g_3}(u_1, z_1 + \zeta_2) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle \\ &= \langle w'_4, \mathcal{Y}_1(Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0) w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle. \end{aligned} \quad (9.25)$$

Therefore, by (9.22), on the region  $O_7$  given by the union of connected components of

$$\{(z_1, \zeta_1, \zeta_2) \in O \mid 0 < |z_1 - \zeta_0| + |\zeta_2| < |\zeta_1|\}$$

which has nonempty intersection with

$$\left\{ (z_1, \zeta_1, \zeta_2) \in O_3 \mid \left| \arg(z_1 + \zeta_2) - \arg(\zeta_1) \right| < \frac{\pi}{2} \right\},$$

we have

$$\langle w'_4, \mathcal{Y}_1(Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0) w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle = h^e(z_1, \zeta_1, \zeta_2) \quad (9.26)$$

Actually  $O_7$  can be characterized by inequalities as the following.

$$O_7 = \left\{ (z_1, \zeta_1, \zeta_2) \in O \mid \begin{array}{l} 0 < |z_1 - \zeta_0| + |\zeta_2| < |\zeta_1|, (\zeta_1, \zeta_2) \in \tilde{U}_1, \\ \left| \arg(z_1 + \zeta_2) - \arg(\zeta_1) \right| < \frac{\pi}{2}, \left| \arg(z_1) - \arg(z_1 + \zeta_2) \right| < \frac{\pi}{2}. \end{array} \right\}$$

Therefore, by (9.17) and (9.26), on the region  $O_8$  given by the union of connected components of

$$\{(z_1, \zeta_1, \zeta_2) \in O \mid 0 < |z_1 - \zeta_0| < |\zeta_0|, |z_1 - \zeta_0| + |\zeta_0| < |\zeta_2|\}$$

which have nonempty intersection with

$$\{(z_1, \zeta_1, \zeta_2) \in O_7 \mid (\zeta_1, \zeta_2) \in \tilde{U}\},$$

we have

$$\begin{aligned}
& \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} (z_1 - \zeta_0)^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\
& \qquad \qquad \qquad \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1 - \zeta_0))^k \\
& = h^e(z_1, \zeta_1, \zeta_2). \tag{9.27}
\end{aligned}$$

Actually  $O_8$  can be characterized by inequalities as the following.

$$\begin{aligned}
& O_8 \\
& = \left\{ \begin{array}{l} (z_1, \zeta_1, \zeta_2) \left| \begin{array}{l} 0 < |z_1 - \zeta_0| < |\zeta_0|, \quad |z_1 - \zeta_0| + |\zeta_0| < |\zeta_2|, \quad (\zeta_1, \zeta_2) \in \tilde{U}_0, \\ \in O \left| \begin{array}{l} \frac{3\pi}{2} < \arg(z_1) - \arg(\zeta_0) < 2\pi, \quad |\arg(z_1 + \zeta_2) - \arg(\zeta_1)| < \frac{\pi}{2}. \end{array} \right. \end{array} \right. \right\}, \\
& \quad \text{if } \tilde{U} = U_3, \\
& = \left\{ \begin{array}{l} (z_1, \zeta_1, \zeta_2) \left| \begin{array}{l} 0 < |z_1 - \zeta_0| < |\zeta_0|, \quad |z_1 - \zeta_0| + |\zeta_0| < |\zeta_2|, \quad (\zeta_1, \zeta_2) \in \tilde{U}_0, \\ \in O \left| \begin{array}{l} |\arg(z_1) - \arg(\zeta_0)| < \frac{\pi}{2}, \quad |\arg(z_1 + \zeta_2) - \arg(\zeta_1)| < \frac{\pi}{2}. \end{array} \right. \end{array} \right. \right\}, \\
& \quad \text{if } \tilde{U} = U_1 \text{ or } U_5. \tag{9.28}
\end{aligned}$$

*Case 1:*  $\tilde{U} = U_1$ . In this case, we know that  $O_5 \cap O_8$  has nonempty interior. Therefore on  $O_5 \cap O_8$  we have

$$\begin{aligned}
& \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} b_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} z_1^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\
& \qquad \qquad \qquad (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k \\
& = h^e(z_1, \zeta_1, \zeta_2) \\
& = \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} (z_1 - \zeta_0)^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\
& \qquad \qquad \qquad \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1 - \zeta_0))^k,
\end{aligned}$$

which implies that for any  $n \in \mathbb{C}$ , on  $O_5 \cap O_8$  we have

$$\begin{aligned}
& \sum_{m \in \mathbb{C}} \sum_{i, j, k \in \mathbb{N}} b_{m, n + \text{wt } w_3 - \text{wt } w'_4 - 1; i, j, k} (u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n - \text{wt } w_3 + \text{wt } w'_4} z_1^{-m-1} \\
& \quad \zeta_0^{n - \text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1 + m + 1} (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k \\
& = \sum_{m \in \mathbb{C}} \sum_{i, j, k \in \mathbb{N}} c_{m, n + \text{wt } w_3 - \text{wt } w'_4 - 1; i, j, k} (u_1; w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j \\
& \quad \cdot (\log(z_1 - \zeta_0))^k \zeta_0^{n - \text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1 + m + 1} \zeta_2^{-n - \text{wt } w_3 + \text{wt } w'_4} (z_1 - \zeta_0)^{-m-1}.
\end{aligned} \tag{9.29}$$

Notice that when  $\tilde{U} = U_1$ , we have  $-\frac{3\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_0) < -\frac{\pi}{2}$  for any  $(z_1, \zeta_1, \zeta_2) \in O_5$ . In other words,  $O_5 \cap O_8$  is contained in the region given by  $(z_1, \zeta_1, \zeta_2) \in \mathbb{C}^3$  such that  $z_1 \in \Omega_{0,0,1}^{(1)}(\zeta_0)$ . On the region given by  $(z_1, \zeta_1, \zeta_2) \in O_5 \cap O_8$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , by (9.13), (9.14), and (9.29), we know that

$$\begin{aligned}
& \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) = h_{(n)}^e(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) \\
& = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2).
\end{aligned} \tag{9.30}$$

Since both sides of (9.30) are single valued analytic on the region given by  $(\zeta_1, \zeta_2) \in \tilde{U}$ ,  $z_1 \in \Omega_{0,0,1}^{(1)}(\zeta_0) \cap \Omega_{0,1,0}^{(1)}(\zeta_0)$ , and are equal on a nonempty open sub-region given by  $(z_1, \zeta_1, \zeta_2) \in O_5 \cap O_8$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , we know that (9.30) holds on the whole region  $\{(z_1, \zeta_1, \zeta_2) \in O \mid (\zeta_1, \zeta_2) \in \tilde{U}, z_1 \in \Omega_{0,0,1}^{(1)}(\zeta_0) \cap \Omega_{0,1,0}^{(1)}(\zeta_0)\}$ .

Together with the final step below, we proved that for any  $(\zeta_1, \zeta_2) \in \tilde{U}$  and  $n \in \mathbb{C}$ , the function  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)$  is  $P(\zeta_0)$ -compatible with respect to  $W_1$  and  $W_2$ . In particular, since  $(\tilde{z}_1, \tilde{z}_2) \in \tilde{U}$ , we know that  $\lambda_n^{(2)}(w_3, w'_4)$  is  $P(\tilde{z}_0)$ -compatible with respect to  $W_1$  and  $W_2$ . In other words,  $\lambda_n^{(2)}(w_3, w'_4) \in \text{COMP}_{P(\tilde{z}_0)}(W_1, W_2)$ .

*Case 2:*  $\tilde{U} = U_3$ . We also know that  $O_5 \cap O_8$  has nonempty interior. Therefore on  $O_5 \cap O_8$ , we still have (9.29). However, when  $\tilde{U} = U_3$ , we have  $\frac{\pi}{2} < \arg(z_1 - \zeta_0) - \arg(\zeta_0) < \frac{3\pi}{2}$  for any  $(z_1, \zeta_1, \zeta_2) \in O_5$ . Therefore in this case, the region given by

$(z_1, \zeta_1, \zeta_2) \in O_5 \cap O_8$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$  is a sub-region of the region given by  $(\zeta_1, \zeta_2) \in \tilde{U}$ ,  $z_1 \in \Omega_{0,0,1}^{(2)}(\zeta_0) \cap \Omega_{0,1,0}^{(1)}(\zeta_0)$ . On the region given by  $(z_1, \zeta_1, \zeta_2) \in O_5 \cap O_8$ ,  $(\zeta_1, \zeta_2) \in \tilde{U}$ , we have

$$\begin{aligned} & \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1)w_2) = h_{(n)}^{b_{1,\zeta_0}^{-1}}(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) \\ & = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0)w_1 \otimes w_2). \end{aligned} \quad (9.31)$$

Similar as *Case 1*, (9.31) holds on the region given by  $(\zeta_1, \zeta_2) \in \tilde{U}$ ,  $z_1 \in \Omega_{0,0,1}^{(2)}(\zeta_0) \cap \Omega_{0,1,0}^{(1)}(\zeta_0)$ .

Together with the final step below, by Remark 7.0.25, we proved that for any  $(\zeta_1, \zeta_2) \in \tilde{U}$  and  $n \in \mathbb{C}$ , the function  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)$  is  $P(\zeta_0)$ -compatible with respect to  $\phi_{g_1}(W_1)$  and  $W_2$ . In particular, since  $(z_1, z_2) \in \tilde{U}$ , we know that  $\lambda_n^{(2)}(w_3, w'_4)$  is  $P(z_0)$ -compatible with respect to  $\phi_{g_1}(W_1)$  and  $W_2$ . In other words,  $\lambda_n^{(2)}(w_3, w'_4) \in \text{COMP}_{P(z_0)}(\phi_{g_1}(W_1), W_2)$ .

*Case 3:*  $\tilde{U} = U_5$ . We have  $O_5 \cap O_8 = \emptyset$ . Define  $O_9$  as follows.

$$O_9 = \left\{ \begin{array}{l} (z_1, \zeta_1, \zeta_2) \left| \begin{array}{l} 0 < |z_1 - \zeta_0| < |\zeta_0|, |z_1 - \zeta_0| + |\zeta_0| < |\zeta_2|, (\zeta_1, \zeta_2) \in \tilde{U}, \\ \in O \left| \begin{array}{l} |\arg(z_1) - \arg(\zeta_0)| < \frac{\pi}{2}, \\ -2\pi < \arg(z_1 + \zeta_2) - \arg(\zeta_1) < -\frac{3\pi}{2}. \end{array} \right. \end{array} \right. \end{array} \right\}$$

The fact that (9.27) holds on the region  $O_8$  implies that on the region  $O_9$ , we have

$$\begin{aligned} & \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} (z_1 - \zeta_0)^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\ & \quad \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1 - \zeta_0))^k \\ & = h^b(z_1, \zeta_1, \zeta_2), \end{aligned} \quad (9.32)$$

where  $b$  is the group element in  $\pi_1(F_{1,2}(\mathbb{C} - \{0\}))$  represented by the loop  $\gamma$  defined

by

$$\gamma(t) = \begin{cases} \left( 3 - \frac{1}{4}t + 4t, 3 - \frac{1}{2}t, 3e^{\frac{\pi}{6}t} \right), & t \in [0, 1/4] \\ \left( 7 + \frac{3}{4}t - 16t, 3 - \frac{1}{2}t, 3e^{\frac{\pi}{6}t} \right), & t \in [1/4, 1/2] \\ \left( -1 + \frac{11}{4}t - 4t, 3 - \frac{1}{2}t, 3e^{\frac{\pi}{6}t} \right), & t \in [1/2, 3/4] \\ \left( -13 - \frac{1}{4}t + 16t, 3 - \frac{1}{2}t, 3e^{\frac{\pi}{6}t} \right). & t \in [3/4, 1] \end{cases}$$

(See Remark8.0.3)

However, we already know that  $z_1 = -\zeta_2$  is just a removable singularity of the multivalued analytic function. Note that  $O_5 \cap O_9$  is a region with nonempty interior contained in the region given by  $(\zeta_1, \zeta_2) \in \tilde{U}$ ,  $z_1 \in \Omega_{0,0,1}^{(1)}(\zeta_0) \cap \Omega_{0,1,0}^{(1)}(\zeta_0)$ . On  $O_5 \cap O_9$ , we have

$$\begin{aligned} & \sum_{m,n \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-1} (z_1 - \zeta_0)^{-m-1} \zeta_0^{m+n+2+\Delta_1} \\ & \quad \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1 - \zeta_0))^k, \\ & = h^b(z_1, \zeta_1, \zeta_2) \\ & = \sum_{n \in \mathbb{C}} \sum_{i \in \mathbb{N}} \zeta_2^{-n-1} (\log(\zeta_2))^i \\ & \quad b. \left( \sum_{m \in \mathbb{C}} \sum_{j,k \in \mathbb{N}} b_{m,n;i,j,k}(u_1; w_1, w_2, w_3, w'_4) z_1^{-m-1} \zeta_0^{m+n+2+\Delta_1} (\log(\zeta_0))^j (\log(z_1))^k \right) \end{aligned}$$

Therefore for any  $n \in \mathbb{C}$ , we have

$$\begin{aligned} & \sum_{m \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} c_{m,n+wt w_3-wt w'_4-1;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j \\ & \quad \cdot (\log(z_1 - \zeta_0))^k \zeta_0^{n-wt w_1-wt w_2-wt w_1+m+1} \zeta_2^{-n-wt w_3+wt w'_4} (z_1 - \zeta_0)^{-m-1}. \\ & = b. \left( \sum_{m \in \mathbb{C}} \sum_{i,j,k \in \mathbb{N}} b_{m,n+wt w_3-wt w'_4-1;i,j,k}(u_1; w_1, w_2, w_3, w'_4) \zeta_2^{-n-wt w_3+wt w'_4} z_1^{-m-1}. \right) \end{aligned}$$

$$\zeta_0^{n-\text{wt } w_1 - \text{wt } w_2 - \text{wt } u_1 + m + 1} (\log(\zeta_2))^i (\log(\zeta_0))^j (\log(z_1))^k$$

Therefore, on the region  $O_5 \cap O_9$ , we have

$$\begin{aligned} & b. \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1) w_2) \right) = b. \left( h_{(n)}^e(\zeta_0, \zeta_2, z_1; u_1, w_1, w_2) \right) \\ & = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) (Y_{W_1}^{g_1}(u_1, z_1 - \zeta_0) w_1 \otimes w_2). \end{aligned} \quad (9.33)$$

Together with the final step below, by Remark 7.0.25, we proved that for any  $(\zeta_1, \zeta_2) \in \tilde{U}$  and  $n \in \mathbb{C}$ , the function  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)$  is  $P(\zeta_0)$ -compatible with respect to  $\phi_{g_1}(W_1)$  and  $W_2$ . In particular, since  $(\tilde{z}_1, \tilde{z}_2) \in \tilde{U}$ , we know that  $\lambda_n^{(2)}(w_3, w'_4)$  is  $P(\tilde{z}_0)$ -compatible with respect to  $\phi_{g_1}(W_1)$  and  $W_2$ . In other words,  $\lambda_n^{(2)}(w_3, w'_4) \in \text{COMP}_{P(\tilde{z}_0)}(\phi_{g_1}(W_1), W_2)$ .

The next step is to prove that

$$\lambda_n^{(2)}(w_1 \otimes Y_{W_2}^{g_2}(u_1, z_1) \cdots Y_{W_2}^{g_2}(u_l, z_l) w_2) \quad (9.34)$$

is absolutely convergent, on the region given by  $0 < |z_l| < \dots < |z_1| < |\tilde{z}_0|$ , to a multivalued analytic function defined on  $(z_1, \dots, z_l) \in F_l(\mathbb{C} - \{0, \tilde{z}_0\})$  satisfying Condition 3.0.13 for  $(l, 0, \{0, \tilde{z}_0\})$ . By using the argument essentially same as the proof of Lemma 3.0.37 and Lemma 7.0.6, we can show that on the region given by  $0 < |z_l| < \dots < |z_1| < |\zeta_2| < |\zeta_1|$

$$\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(Y_{W_2}^{g_2}(u_1, z_1) \cdots Y_{W_2}^{g_2}(u_l, z_l) w_2, \zeta_2) w_3 \rangle$$

is absolutely convergent to

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, l+2}(z_1 + \zeta_2, \dots, z_l + \zeta_2, \zeta_1, \zeta_2; u_1, \dots, u_l, w_1, w_2, w_3, w'_4) \\ & = f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), l+2}(z_1, \dots, z_l, \zeta_0, -\zeta_2; u_1, \dots, u_l, w_1, w_2, w_3, e^{\zeta_2 L(1)} w'_4) \end{aligned} \quad (9.35)$$

Therefore  $\zeta_2 = \infty$  is a regular singularity of (9.35) for  $(z_1, \dots, z_l, \zeta_0) \in F_{l+1}(\mathbb{C}^\times)$ . We know that (9.34) is equal to the sum of coefficients of  $\zeta_2^{-n-wt w_3+wt w_4} \log(\zeta_2)^k$  over  $k \in \mathbb{N}$  in the corresponding expansion with substitution  $\zeta_0 = \tilde{z}_0$ ,  $\zeta_2 = \tilde{z}_2$  (the substitution  $\zeta_2 = \tilde{z}_2$  is involved only because of the polynomial  $e^{\zeta_2 L(1)w'_4}$ ). Therefore, by part (e) of Condition 3.0.13, we know that the multivalued function that (9.34) converges to, denoted by

$$f_l(z_1, \dots, z_l; u_1, \dots, u_l, w_1, w_2; \lambda_n^{(2)}),$$

satisfies Condition 3.0.13 for  $(l, 0, \{0, \tilde{z}_0\})$ .

The final step is to prove that the multivalued function

$$f_l \left( \frac{z_1 \tilde{z}_0}{z_{l+1}}, \dots, \frac{z_l \tilde{z}_0}{z_{l+1}}; u_1, \dots, u_l, w_1, w_2; \lambda_n^{(2)} \right) \quad (9.36)$$

satisfies Condition  $(l, 1, \{0\})$ . We know that on the region given by  $0 < |z_l| < |z_{l-1}| < \dots < |z_1| < |z_{l+1}|$ , the series

$$\lambda_n^{(2)} \left( w_1 \otimes Y_{W_2}^{g_2} \left( u_1, \frac{z_1 \tilde{z}_0}{z_{l+1}} \right) \dots Y_{W_2}^{g_2} \left( u_l, \frac{z_l \tilde{z}_0}{z_{l+1}} \right) w_2 \right) \quad (9.37)$$

is absolutely convergent to (9.36). Consider

$$\left\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2 \left( Y_{W_2}^{g_2} \left( u_1, \frac{z_1 \zeta_0}{z_{l+1}} \right) \dots Y_{W_2}^{g_2} \left( u_l, \frac{z_l \zeta_0}{z_{l+1}} \right) w_2, \zeta_2 \right) w_3 \right\rangle, \quad (9.38)$$

which is absolutely convergent, on the region given by  $0 < \left| \frac{z_l \zeta_0}{z_{l+1}} \right| < \dots < \left| \frac{z_1 \zeta_0}{z_{l+1}} \right| < |\zeta_2| < |\zeta_1|$ , to

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, l+2} \left( \frac{z_1 \zeta_0}{z_{l+1}} + \zeta_2, \dots, \frac{z_l \zeta_0}{z_{l+1}} + \zeta_2, \zeta_1, \zeta_2; u_1, \dots, u_l, w_1, w_2, w_3, w'_4 \right) \\ &= f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), l+2} \left( \frac{z_1 \zeta_0}{z_{l+1}}, \dots, \frac{z_l \zeta_0}{z_{l+1}}, \zeta_0, -\zeta_2; u_1, \dots, u_l, w_1, w_2, w_3, e^{\zeta_2 L(1)w'_4} \right) \end{aligned}$$

$$\begin{aligned}
&= f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), l+2} \left( z_1, \dots, z_l, z_{l+1}, X; (\zeta_0^{-1})^{-L(0)} z_{l+1}^{-L(0)} u_1, \dots, (\zeta_0^{-1})^{-L(0)} z_{l+1}^{-L(0)} u_l, \right. \\
&\quad \left. (\zeta_0^{-1})^{-L(0)} z_{l+1}^{-L(0)} w_1, (\zeta_0^{-1})^{-L(0)} z_{l+1}^{-L(0)} w_2, (\zeta_0^{-1})^{-L(0)} z_{l+1}^{-L(0)} w_3, (\zeta_0^{-1})^{L(0)} z_{l+1}^{L(0)} e^{\zeta_2 L(1)} w'_4 \right), \tag{9.39}
\end{aligned}$$

where  $X = -\frac{\zeta_2 z_{l+1}}{\zeta_0}$ . Notice that  $(\zeta_0^{-1})^{-L(0)} z_{l+1}^{-L(0)} w_1$  is a finite sum of elements of the form  $\zeta_0^a z_{l+1}^b \log(\zeta_0)^{k_1} \log(z_{l+1})^{k_2} w_1$  for  $a, b \in \mathbb{C}$ ,  $k_1, k_2 \in \mathbb{N}$ . It is similar for the Virasoro action on  $u_1, \dots, u_l$ ,  $w_2$ ,  $w_3$ , and  $w'_4$  in (9.39). They have no influence on the regularity of any singularity.

Since  $X = \infty$  is a regular singularity of (9.39) for  $(z_1, \dots, z_l, z_{l+1}) \in F_{l+1}(\mathbb{C}^\times)$ , applying part (e) of Condition 3.0.13 for  $(l, 2, \{0\})$  to (9.39), we know that (9.36) satisfies Condition 3.0.13 for  $(l, 1, \{0\})$ . Indeed, (9.36) is a coefficient of the expansion of (9.39) corresponding to the regular singularity  $\zeta_2 = \infty$ . Multiplying a certain power of  $(-z_{l+1}/\zeta_0)$  to a coefficient of the expansion corresponding to  $X = \infty$ , one would obtain the coefficient of the expansion corresponding to  $\zeta_2 = \infty$ .

□

From (9.2), we know that near the point  $(\zeta_1, \zeta_2) = (\tilde{z}_1, \tilde{z}_2) \in \widehat{M}_0^2$ , we have

$$\begin{aligned}
&\langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle \\
&= \sum_{n \in \mathbb{C}} \sum_{i, j \in \mathbb{N}} \tilde{a}_{n; i, j}(w_1, w_2, w_3, w'_4) e^{(-n-1)l_0(\zeta_2)} (l_0(\zeta_2))^i e^{(\Delta+n+1)l_0(\zeta_1-\zeta_2)} (l_0(\zeta_1-\zeta_2))^j. \tag{9.40}
\end{aligned}$$

For  $n \in \mathbb{C}$ ,  $i, j \in \mathbb{N}$ ,  $w'_4 \in W'_4$ ,  $w_3 \in W_3$ , define

$$\begin{aligned}
&\beta_{n; i, j}(w'_4, w_3; \zeta_0)(w_1 \otimes w_2) = \\
&= \tilde{a}_{n; i, j}(e^{-L(0)_s l_0(\zeta_0)} w_1, e^{-L(0)_s l_0(\zeta_0)} w_2, e^{-L(0)_s l_0(\zeta_0)} w_3, e^{L(0)_s l_0(\zeta_0)} w'_4). \tag{9.41}
\end{aligned}$$

We have

$$\lambda_n^{(2)}(w_1 \otimes w_2) = \left( \sum_{i,j \in \mathbb{N}} \beta_{n-\text{wt } w'_4 + \text{wt } w_3 - 1; i, j}(w'_4, w_3; \zeta_0)(w_1 \otimes w_2) \cdot e^{(n-\text{wt } w'_4 + \text{wt } w_3)l_0(\zeta_0)} e^{(-n+\text{wt } w'_4 - \text{wt } w_3)l_0(\zeta_2)} (\log(\zeta_2))^i (\log(\zeta_0))^j \right) \Big|_{\zeta_1 = \tilde{z}_1, \zeta_2 = \tilde{z}_2}. \quad (9.42)$$

Similar to theorem 9.0.1, it is not hard to prove that  $\beta_{n; i, j}(w'_4, w_3; \zeta_0)$  is  $P(\zeta_0)$ -compatible.

Since

$$\begin{aligned} & f_1^e(z_1; v, w_1, w_2; \beta_{n; i, j}(w'_4, w_3; \zeta_0)) \\ &= \sum_{m \in \mathbb{C}} \sum_{k \in \mathbb{N}} \tilde{a}_{n; i, j}(v_{-m-1, k}(w_1), w_2, w_3, w'_4) e^{(-m-\text{wt } v + \Delta)l_0(\zeta_0)} e^{ml_0(z-\zeta_0)} (l_0(z-\zeta_0))^k \end{aligned} \quad (9.43)$$

Since  $(\zeta_1 - \zeta_2, \zeta_2) = (0, \infty)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 2}(\zeta_1, \zeta_2; w_1, w_2, w_3, w'_4)$ , by the definition of  $\tilde{\lambda}_n^{(2)}$ , we easily have the following property.

**Proposition 9.0.3.** *For any  $w_3 \in W_3$  and  $w'_4 \in W'_4$ , there exists  $N' \in \mathbb{N}$ , such that*

$$\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) = 0, \text{ for any } n \in \mathbb{C}, \text{ with } \Re(n) < -N', \text{ and } (\zeta_1, \zeta_2) \in F_2(\mathbb{C}^\times). \quad (9.44)$$

We denote the minimal  $N'$  such that (9.44) holds by  $N(w_3, w'_4)$ .

**Theorem 9.0.4.** *Suppose that  $\{N(w_3, w'_4) | w_3 \in W_3, w'_4 \in W'_4\}$  (see Proposition 9.0.3) has an upper bound  $N \in \mathbb{N}$ . In other words, there exists  $N \in \mathbb{N}$  such that*

$$\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) = 0,$$

$$\text{for any } n \in \mathbb{C} \text{ with } \Re(n) < -N, w_3 \in W_3, w'_4 \in W'_4, \text{ and } (\zeta_1, \zeta_2) \in F_2(\mathbb{C}^\times).$$

Then, the linear functional  $\lambda_n^{(2)}(w_3, w'_4) = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \tilde{z}_1, \tilde{z}_2) \in (W_1 \otimes W_2)^*$  is  $P(\tilde{z}_0)$ -

locally-grading-restricted for any  $n \in \mathbb{C}$ .

*Proof.* Since  $W_1, W_2, W_3$ , and  $W'_4$ , are generalized grading restricted untwisted  $V^G$ -modules, and the Virasoro action  $L(n)$  are induced by the conformal element  $\omega \in V^G$ , we know that the argument from (11.31) to (11.37) in [HLZ4] still hold. In particular, there exists  $L \in \mathbb{Z}_+$  independent of  $\zeta_1, \zeta_2 \in \mathbb{C}$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$ , such that

$$\begin{aligned} & (L_{P(\zeta_0)}(0) - \text{wt } w'_4 + \text{wt } w_3 - n - 1)^L (\beta_{n;i,j}(w'_4, w_3)) = 0, \\ & \text{or equivalently, } \text{wt } \beta_{n;i,j}(w'_4, w_3) = n + \text{wt } w'_4 - \text{wt } w_3 + 1 \end{aligned} \quad (9.45)$$

Now, we want to study  $\left(Y_{P(\zeta_0)}^{(g_1 g_2)^{-1}}\right)^o(v, x)(\beta_{n;i,j}(w'_4, w_3))$ . Since  $\beta_{n;i,j}(w'_4, w_3)$  is  $P(\zeta_0)$ -compatible, we know that there exists the function  $f_1(z_1; v, w_1, w_2; \beta_{n;i,j}(w'_4, w_3))$  on  $z_1 \in M^1(0, \zeta_0)$  as in the definition of  $P(\zeta_0)$ -compatibility, and a preferred branch  $f_1^e(z_1; v, w_1, w_2; \beta_{n;i,j}(w'_4, w_3))$  which is single valued analytic on the region given by  $z_1 \in M_0^1(0, \zeta_0)$ . There exist  $a_{i',m,j';n,i,j} = a_{i',m,j';n,i,j}(v, w_1, w_2; \beta_{n;i,j}(w'_4, w_3)) \in \mathbb{C}$ , such that on the region given by  $|z_1| > |\zeta_0|$ ,  $|\arg(z_1 - \zeta_0) - \arg(z_1)| < \frac{\pi}{2}$ , we have the expansion

$$f_1^e(z_1; v, w_1, w_2; \beta_{n;i,j}(w'_4, w_3)) = \sum_{i',j'=0}^M \sum_{m \in \mathbb{N}} a_{i',m,j';n,i,j} e^{(r_{i'}-m)l_0(z_1)} (l_0(z_1))^{j'}. \quad (9.46)$$

$$\begin{aligned} & \sum_{n \in \mathbb{C}} \sum_{i,j \in \mathbb{N}} \beta_{n;i,j}(w'_4, w_3) (w_1 \otimes w_2) e^{(n+1)l_0(\zeta_0)} e^{(-n-1)l_0(-\zeta_2)} (l_0(\zeta_2))^i (l_0(\zeta_0))^j \\ & = \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle \end{aligned} \quad (9.47)$$

We define

$$\left(Y_{P(z)}^{(g_1 g_2)^{-1}}(e^{xL(1)}(-x^2)^{L(0)}u, x^{-1})(\lambda)\right) (w_1 \otimes w_2)$$

$$= \sum_{i,j=0}^N \sum_{n \in \mathbb{N}} a_{i,n,j}(u, w_1, w_2; \lambda) x^{r_i - n} (\log x)^j. \quad (9.48)$$

We define  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \in (W_1 \otimes W_2)^*$  by

$$\begin{aligned} & \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2)(w_1 \otimes w_2) = \\ & \sum_{i,j \in \mathbb{N}} \tilde{a}_{n+wt w_3 - wt w'_4 - 1; i, j} (e^{-l_0(\zeta_0)L(0)_s} w_1, e^{-l_0(\zeta_0)L(0)_s} w_2, e^{-l_0(\zeta_0)L(0)_s} w_3, e^{l_0(\zeta_0)L(0)_s} w'_4) \\ & \cdot e^{(n+wt w_3 - wt w'_4)l_0(\zeta_0)} e^{(-n - wt w_3 + wt w'_4)l_0(\zeta_2)} (\log(\zeta_2))^i (\log(\zeta_0))^j. \end{aligned}$$

Therefore,  $\lambda_n^{(2)}(w_3, w'_4) = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \tilde{z}_1, \tilde{z}_2)$ . Let  $\Delta_{v, n_1} = \text{wt}(Y_{n_1, i_1}^o(v)(w'_4)) - \text{wt} w_1 - \text{wt} w_2 - \text{wt} w_3 = \Delta - \text{wt} v + n_1 + 1$ .

For any  $n_1 \in \mathbb{C}$ ,  $i_1 \in \mathbb{N}$ , there exists  $a_{n_1, i_1; k, i'} \in \mathbb{C}$  such that on the region  $0 < |\zeta_2| < |z_1|$ , and (yi ge fujiao budengshi)

$$e^{(-n_1 - 1)l_0(z_1 + \zeta_2)} (l_0(z_1 + \zeta_2))^{i_1} = e^{(-n_1 - 1)l_0(z_1)} \sum_{i'=0}^{i_1} \sum_{k \in \mathbb{N}} a_{n_1, i_1; k, i'} \zeta_2^k z_1^{-k} l_0(z_1)^{i'}.$$

On the region given by  $0 < |\zeta_2| < |\zeta_1| < |z_1 + \zeta_2|$ ,  $|z_1| > 2|\zeta_1|$ ,

$$\begin{aligned} & \langle w'_4, Y_{W_4}^{g_1 g_2 g_3}(v, z_1 + \zeta_2) \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle \\ & = \left\langle \left( Y_{W_4}^{(g_1 g_2 g_3)^{-1}} \right)^o (v, z_1 + \zeta_2) w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \right\rangle \\ & = \sum_{n_1 \in D} \sum_{i_1 \in \mathbb{N}} \langle Y_{n_1, i_1}^o(v)(w'_4), \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle e^{(-n_1 - 1)l_0(z_1 + \zeta_2)} (l_0(z_1 + \zeta_2))^{i_1} \\ & = \sum_{n_1 \in D} \sum_{i_1 \in \mathbb{N}} \sum_{n \in \mathbb{C}} \tilde{\lambda}_n^{(2)}(w_3, Y_{n_1, i_1}^o(v)(w'_4); \zeta_1, \zeta_2) e^{(-n_1 - 1)l_0(z_1 + \zeta_2)} (l_0(z_1 + \zeta_2))^{i_1} \\ & = \sum_{n_1 \in D} \sum_{i_1 \in \mathbb{N}} \sum_{n \in \mathbb{C}} \tilde{\lambda}_n^{(2)}(w_3, Y_{n_1, i_1}^o(v)(w'_4); \zeta_1, \zeta_2) (w_1 \otimes w_2) \cdot \\ & \quad \left( e^{(-n_1 - 1)l_0(z_1)} \sum_{i'=0}^{i_1} \sum_{k \in \mathbb{N}} a_{n_1, i_1; k, i'} \zeta_2^k z_1^{-k} l_0(z_1)^{i'} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\tilde{n} \in \mathbb{C}} \sum_{i' \in \mathbb{N}} \left( \sum_{n \in \mathbb{C}} \sum_{i_1 \geq i'} \sum_{\substack{n_1 \in D, k \in \mathbb{N}, \\ -n_1 - 1 - k = \tilde{n}}} a_{n_1, i_1; k, i'} \zeta_2^k \cdot \right. \\
&\quad \left. \tilde{\lambda}_n^{(2)}(w_3, Y_{n_1, i_1}^o(v)(w'_4); \zeta_1, \zeta_2)(w_1 \otimes w_2) \right) e^{\tilde{n} l_0(z_1)} (l_0(z_1))^{i'} \\
&= \sum_{\tilde{n} \in \mathbb{C}} \sum_{i' \in \mathbb{N}} \left[ \sum_{n \in \mathbb{C}} \sum_{i_1 \geq i'} \sum_{\substack{n_1 \in D, k \in \mathbb{N}, \\ -n_1 - 1 - k = \tilde{n}}} a_{n_1, i_1; k, i'} \zeta_2^k \cdot \right. \\
&\quad \left( \sum_{i, j \in \mathbb{N}} \tilde{a}_{n - \tilde{n} + \text{wt } w_3 - \text{wt } w'_4 - n_1 - 2; i, j}(w_1, w_2, w_3, Y_{n_1, i_1}^o(v)(w'_4)) e^{(n - \text{wt } v - \tilde{n} - \text{wt } w_1 - \text{wt } w_2) l_0(\zeta_0)} \right. \\
&\quad \left. \left. e^{(-n + \tilde{n} + n_1 - \text{wt } w_3 + \text{wt } w'_4 + 1) l_0(\zeta_2)} (l_0(\zeta_2))^i (l_0(\zeta_0))^j \right) \right] e^{\tilde{n} l_0(z_1)} (l_0(z_1))^{i'} \\
&= \sum_{\tilde{n} \in \mathbb{C}} \sum_{i' \in \mathbb{N}} \left[ \sum_{n \in \mathbb{C}} e^{(-n - \text{wt } w_3 + \text{wt } w'_4) l_0(\zeta_2)} e^{(n - \text{wt } v - \tilde{n} - \text{wt } w_1 - \text{wt } w_2) l_0(\zeta_0)} \cdot \right. \\
&\quad \sum_{\substack{i_1 \geq i' \\ n_1 \in D, k \in \mathbb{N}, \\ -n_1 - 1 - k = \tilde{n}}} a_{n_1, i_1; k, i'} \left( \sum_{i, j \in \mathbb{N}} \tilde{a}_{n - \tilde{n} + \text{wt } w_3 - \text{wt } w'_4 - n_1 - 2; i, j}(w_1, w_2, w_3, Y_{n_1, i_1}^o(v)(w'_4)) \right. \\
&\quad \left. \left. (l_0(\zeta_2))^i (l_0(\zeta_0))^j \right) \right] e^{\tilde{n} l_0(z_1)} (l_0(z_1))^{i'} \\
&= \sum_{\tilde{n} \in \mathbb{C}} \sum_{n \in \mathbb{C}} \sum_{i' \in \mathbb{N}} \sum_{i, j \in \mathbb{N}} \\
&\quad \left( \sum_{\substack{i_1 \geq i' \\ n_1 \in D, k \in \mathbb{N}, \\ -n_1 - 1 - k = \tilde{n}}} a_{n_1, i_1; k, i'} \cdot \tilde{a}_{n - \tilde{n} + \text{wt } w_3 - \text{wt } w'_4 - n_1 - 2; i, j}(w_1, w_2, w_3, Y_{n_1, i_1}^o(v)(w'_4)) \right) \\
&\quad \cdot e^{(-n - \text{wt } w_3 + \text{wt } w'_4) l_0(\zeta_2)} e^{(n - \text{wt } v - \tilde{n} - \text{wt } w_1 - \text{wt } w_2) l_0(\zeta_0)} e^{\tilde{n} l_0(z_1)} (l_0(z_1))^{i'} (l_0(\zeta_2))^i (l_0(\zeta_0))^j
\end{aligned} \tag{9.49}$$

The right-hand side is absolutely convergent because  $(\frac{z_1}{\zeta_2}, \frac{\zeta_0}{\zeta_2}) = (\infty, 0)$  is a regular singularity of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 1}(\frac{\zeta_0}{\zeta_2} + 1, 1, \frac{z_1}{\zeta_2}; v; w_1, w_2, w_3, w'_4)$ .

$$\left( Y_{P(\zeta_0)}^{(g_1 g_2)^{-1}} \right)^o(v, z_1) \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right) (w_1 \otimes w_2)$$

$$\begin{aligned}
&= f_1^e \left( z_1; v, w_1, w_2; \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right) \\
&= \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \left( Y_{W_1}^{g_1}(v, z_1 - \zeta_0) w_1 \otimes w_2 \right) \\
&= \sum_{i,j \in \mathbb{N}} \tilde{a}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j} \left( e^{-l_0(\zeta_0)L(0)_s} Y_{W_1}^{g_1}(v, z_1 - \zeta_0) w_1, e^{-l_0(\zeta_0)L(0)_s} w_2, e^{-l_0(\zeta_0)L(0)_s} w_3, \right. \\
&\quad \left. e^{l_0(\zeta_0)L(0)_s} w'_4 \right) \cdot e^{(n+\text{wt } w_3 - \text{wt } w'_4)l_0(\zeta_0)} e^{(-n-\text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)} (\log(\zeta_2))^i (\log(\zeta_0))^j \\
&= \sum_{i,j, i' \in \mathbb{N}} \sum_{m \in \mathbb{C}} \tilde{a}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j} \left( Y_{m, i'}(v) w_1, w_2, w_3, w'_4 \right) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (l_0(z_1 - \zeta_0))^{i'} \\
&\quad \cdot e^{(n-\text{wt } w_1 - \text{wt } w_2 - \text{wt } v + m + 1)l_0(\zeta_0)} e^{(-n-\text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)} e^{(-m-1)l_0(z_1 - \zeta_0)} \\
&= \sum_{i,j, i' \in \mathbb{N}} \sum_{\tilde{n} \in \mathbb{C}} \tilde{b}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j} \left( Y_{m, i'}(v) w_1, w_2, w_3, w'_4 \right) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j (l_0(z_1))^{i'} \\
&\quad \cdot e^{(n-\text{wt } w_1 - \text{wt } w_2 - \text{wt } v - \tilde{n})l_0(\zeta_0)} e^{(-n-\text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)} e^{\tilde{n}l_0(z_1)},
\end{aligned}$$

where  $\tilde{b}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j}$  are some complex numbers, whose existence is because of lemma 4.1 in *a convergence lemma*, and the fact that  $z_1 = \infty$  is a regular singularity of

$$f_1^e \left( z_1; v, w_1, w_2; \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right).$$

Therefore, we know that

$$\begin{aligned}
&\left( Y_{P(\zeta_0)}^{(g_1 g_2)^{-1}} \right)_{-\tilde{n}-1, i'}^o(v) \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right) \\
&= \sum_{i, j \in \mathbb{N}} \tilde{b}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j} \left( Y_{m, i'}(v) w_1, w_2, w_3, w'_4 \right) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j \\
&\quad \cdot e^{(n-\text{wt } w_1 - \text{wt } w_2 - \text{wt } v - \tilde{n})l_0(\zeta_0)} e^{(-n-\text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)}. \tag{9.50}
\end{aligned}$$

Since the right-hand side of (9.49) is equal to

$$\begin{aligned}
&\sum_{n, \tilde{n} \in \mathbb{C}} \sum_{i' \in \mathbb{N}} \left( Y_{P(\zeta_0)}^{(g_1 g_2)^{-1}} \right)_{-\tilde{n}-1, i'}^o(v) \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right) (w_1 \otimes w_2) \cdot e^{\tilde{n}l_0(z_1)} (l_0(z_1))^{i'} \\
&= \sum_{n, \tilde{n} \in \mathbb{C}} \sum_{i, j, i' \in \mathbb{N}} \tilde{b}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j} \left( Y_{m, i'}(v) w_1, w_2, w_3, w'_4 \right) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j
\end{aligned}$$

$$\cdot e^{(n-\text{wt } w_1 - \text{wt } w_2 - \text{wt } v - \tilde{n})l_0(\zeta_0)} e^{(-n - \text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)} e^{\tilde{n}l_0(z_1)} (l_0(z_1))^{i'}. \quad (9.51)$$

Compare the coefficient of the right-hand side of (9.49) and (9.51), we know that

$$\begin{aligned} & \left( Y_{P(\zeta_0)}^{(g_1 g_2)^{-1}} \right)_{-\tilde{n}-1, i'}^o (v) \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right) \\ &= \sum_{i, j \in \mathbb{N}} \tilde{b}_{n+\text{wt } w_3 - \text{wt } w'_4 - 1; i, j} (Y_{m, i'}(v) w_1, w_2, w_3, w'_4) \cdot (\log(\zeta_2))^i (\log(\zeta_0))^j \\ & \quad \cdot e^{(n-\text{wt } w_1 - \text{wt } w_2 - \text{wt } v - \tilde{n})l_0(\zeta_0)} e^{(-n - \text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)} \\ &= \sum_{i, j \in \mathbb{N}} \left( \sum_{i_1 \geq i'} \sum_{\substack{n_1 \in D, k \in \mathbb{N}, \\ -n_1 - 1 - k = \tilde{n}}} a_{n_1, i_1; k, i'} \cdot \tilde{a}_{n - \tilde{n} + \text{wt } w_3 - \text{wt } w'_4 - n_1 - 2; i, j} (w_1, w_2, w_3, Y_{n_1, i_1}^o(v)(w'_4)) \right) \\ & \quad e^{(-n - \text{wt } w_3 + \text{wt } w'_4)l_0(\zeta_2)} e^{(n - \text{wt } v - \tilde{n} - \text{wt } w_1 - \text{wt } w_2)l_0(\zeta_0)} (l_0(\zeta_2))^i (l_0(\zeta_0))^j \\ &= \sum_{i_1 \geq i'} \sum_{\substack{n_1 \in D, k \in \mathbb{N}, \\ -n_1 - 1 - k = \tilde{n}}} a_{n_1, i_1; k, i'} \zeta_2^k \cdot \tilde{\lambda}_{n - \text{wt } v - \tilde{n}}^{(2)}(w_3, Y_{n_1, i_1}^o(v)(w'_4); \zeta_1, \zeta_2) \end{aligned} \quad (9.52)$$

By (9.45), we know that  $\text{wt } \tilde{\lambda}_n(w_3, w'_4; \zeta_1, \zeta_2) = n$  for any  $n \in \mathbb{C}$ . Therefore, by (9.52), we know that

$$\text{wt } Y_{n_1, i_1}^o(v_1) \cdots Y_{n_r, i_r}^o(v_r) \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) \right) = n - \sum_{i=1}^r (\text{wt } v_i - n_i - 1). \quad (9.53)$$

Therefore, we know that

$$W_{\lambda_n^{(2)}(w_3, w'_4)} = \prod_{l \in \mathbb{C}} \left( W_{\lambda_n^{(2)}(w_3, w'_4)} \right)_{[l]} \quad (9.54)$$

Since we have

$$\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) = 0 \text{ if } \Re(n) \leq -N, \quad (9.55)$$

for any  $n \in \mathbb{C}$ , and  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , by (9.52) and the definition of  $W_{\lambda_n^{(2)}(w_3, w'_4)}$ , we

have  $\left(W_{\lambda_n^{(2)}(w_3, w'_4)}\right)_{[l]} = 0$  for any  $l \in \mathbb{C}$  with  $\Re(l) \leq -N$ .

□

Basically, the  $P(z)$  story is finished here.

# Chapter 10

## The $Q(z)$ Part

For any  $z \in \mathbb{C}^\times$ , a  $Q(z)$ -intertwining map of type  $\binom{W'_1}{W'_3 W_2}$  is a linear map  $J$  from  $W'_3 \otimes W_2$  to  $\overline{W'_1} = (W_1)^*$  of the form

$$J(w'_3 \otimes w_2)(w_1) = \langle w'_3, \mathcal{Y}(w_1, z)w_2 \rangle, \quad \text{for all } w_1 \in W_1, w_2 \in W_2, w_3 \in W'_3,$$

where  $\mathcal{Y}$  is a twisted intertwining operator of type  $\binom{W_3}{W_1 W_2}$ . By using the  $\Omega_\pm$  and  $A_\pm$  operator, we have

$$\begin{aligned} \langle w'_3, \mathcal{Y}(w_1, x)w_2 \rangle &= \langle e^{xL(1)}w'_3, \Omega_\pm(\mathcal{Y})(w_2, e^{\mp\pi i}x)w_1 \rangle \\ &= \langle A_\pm(\Omega_\pm(\mathcal{Y})) \left( x^{-2L(0)} e^{(\pm 2 \mp 1)\pi i L(0)} e^{x^{-1}L(1)} w_2, e^{\pm\pi i} x^{-1} \right) e^{xL(1)} w'_3, w_1 \rangle \\ &= \langle \Omega_\pm(A_\pm(\Omega_\pm(\mathcal{Y}))) (e^{xL(1)} w'_3, e^{(\pm 1 \mp 1)\pi i} x^{-1}) x^{-2L(0)} e^{(\pm 2 \mp 1)\pi i L(0)} e^{x^{-1}L(1)} w_2, e^{-x^{-1}L(1)} w_1 \rangle. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\langle e^{-xL(1)}w'_3, \mathcal{Y}(e^{x^{-1}L(1)}w_1, x)e^{-x^{-1}L(1)}e^{(\mp 2 \pm 1)\pi i L(0)}x^{2L(0)}w_2 \rangle \\ &= \langle \Omega_\pm(A_\pm(\Omega_\pm(\mathcal{Y}))) (w'_3, x^{-1}) w_2, w_1 \rangle. \end{aligned} \tag{10.1}$$

Let  $p = p(\tilde{z}_2)$  be the unique integer satisfying  $l_p(\tilde{z}_2^{-1}) = -l_0(\tilde{z}_2)$ . Then, we know that

$$\begin{aligned} & \left\langle e^{-\tilde{z}_2 L(1)} w'_3, \mathcal{Y}(e^{\tilde{z}_2^{-1} L(1)} w_1, \tilde{z}_2) e^{-\tilde{z}_2^{-1} L(1)} e^{(\mp 2 \pm 1) \pi i L(0)} \tilde{z}_2^{2L(0)} w_2 \right\rangle \\ &= \left\langle \Omega_{\pm}(A_{\pm}(\Omega_{\pm}(\mathcal{Y})))^p (w'_3, \tilde{z}_2^{-1}) w_2, w_1 \right\rangle. \end{aligned}$$

When  $\tilde{z}_2 \notin \mathbb{R}_+$ , we have  $p = -1$ , and  $\Omega_{-}(A_{+}(\Omega_{-}(\mathcal{Y})))^p$  is a twisted intertwining operator of type

$$\begin{pmatrix} \phi_{g_1 g_2}(W'_1) \\ \phi_{g_1 g_2^2}(W'_3) \quad \phi_{g_1 g_2^{-1}}(W_2) \end{pmatrix} \quad \text{if } \tilde{z}_2 \notin \mathbb{R}_+, \quad (10.2)$$

For any  $J \in \text{Hom}(W'_3 \otimes W_2, \overline{W'_1})$  and  $z \in \mathbb{C}^\times$ , we define  $B_z(J) \in \text{Hom}(W'_3 \otimes W_2, \overline{W'_1})$  as follows (we identify  $\overline{W'_1}$  with  $(W_1)^*$ ),

$$B_z(J)(w'_3 \otimes w_2)(w_1) = \left\langle e^{z^{-1} L(1)} w_1, J \left( e^{-z L(1)} w'_3 \otimes e^{-z^{-1} L(1)} e^{(\mp 2 \pm 1) \pi i L(0)} z^{2L(0)} w_2 \right) \right\rangle$$

Since  $\Omega_{\pm}$  and  $A_{\pm}$  are bijections between different spaces of intertwining operators, we know that  $J$  is a  $Q(\tilde{z}_2)$ -intertwining map if and only if  $B_{\tilde{z}_2}(J)$  is a  $P(\tilde{z}_2^{-1})$ -intertwining map of the type (10.2), which is equivalent to the fact that

$$\begin{aligned} \mathcal{Y}_{B_{\tilde{z}_2}(J)}(w'_3, x) w_2 &= x^{L(0)} (\tilde{z}_2^{-1})^{-L(0)} B_{\tilde{z}_2}(J) \left( x^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} w'_3 \otimes x^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} w_2 \right) \\ &= x^{L(0)} (\tilde{z}_2^{-1})^{-L(0)} e^{\tilde{z}_2^{-1} L(-1)} J \left( e^{-\tilde{z}_2 L(1)} x^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} w'_3 \otimes \right. \\ & \quad \left. e^{-\tilde{z}_2^{-1} L(1)} e^{(\mp 2 \pm 1) \pi i L(0)} \tilde{z}_2^{2L(0)} x^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} w_2 \right) \end{aligned} \quad (10.3)$$

is a twisted intertwining operator of type (10.2). Let  $r = \mp 2 \pm 1 \in \{-3, -1, 1, 3\}$ . In the following, we take  $r = +2+1 = 3$ .

Now, let  $J \in \text{Hom}(W'_4 \otimes W_3, \overline{W_1 \boxtimes_{P(\tilde{z}_0)} W_2})$  be defined by

$$\pi_n(J(w'_4 \otimes w_3)) = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \tilde{z}_1, \tilde{z}_2) \in (W_1 \boxtimes_{P(\tilde{z}_0)} W_2)_{[n]}, \quad n \in \mathbb{C}. \quad (10.4)$$

When we identify  $\overline{W_1 \boxtimes_{P(\tilde{z}_0)} W_2}$  with  $(W_1 \boxtimes_{P(\tilde{z}_0)} W_2)^*$ , we know that the linear map  $J$  is uniquely determined by

$$J(w'_4 \otimes w_3)(\pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2)) = \tilde{\lambda}_n^{(2)}(w_3, w'_4; \tilde{z}_1, \tilde{z}_2)(w_1 \otimes w_2), \quad (10.5)$$

for any  $w_1 \in W_1$ ,  $w_2 \in W_2$ , and  $n \in \mathbb{C}$ . By (10.6) and (9.2), we know that

$$\begin{aligned} J(w'_4 \otimes w_3)(w_1 \boxtimes_{P(\tilde{z}_0)} w_2) &= \sum_{n \in \mathbb{C}} J(w'_4 \otimes w_3)(\pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2)) \\ &= \langle w'_4, \mathcal{Y}_1(w_1, \tilde{z}_1) \mathcal{Y}_2(w_2, \tilde{z}_2) w_3 \rangle. \end{aligned} \quad (10.6)$$

We want to show that  $J$  is a  $Q(\tilde{z}_2)$ -intertwining map of type

$$\begin{pmatrix} W_1 \boxtimes_{P(\tilde{z}_0)} W_2 \\ W'_4 \ W_3 \end{pmatrix}. \quad (10.7)$$

From the above discussion, it is equivalent to prove that  $\mathcal{Y}_{B_{\tilde{z}_2}(J)}(\cdot, x) \cdot$  is a twisted intertwining operator of the type

$$\begin{pmatrix} \phi_{g_1 g_2 g_3}(W_1 \boxtimes_{P(\tilde{z}_0)} W_2) \\ \phi_{g_1 g_2 g_3^2}(W'_4) \ \phi_{g_1 g_2 g_3^{-1}}(W_3) \end{pmatrix} \quad \text{if } \tilde{z}_2 \notin \mathbb{R}_+.$$

**Theorem 10.0.1.** *For any  $(\tilde{z}_1, \tilde{z}_2) \in U_1$  (See remark 8.0.7) with  $\tilde{z}_2 \notin \mathbb{R}_+$ , the map  $\mathcal{Y}_{B_{\tilde{z}_2}(J)}(\cdot, x) \cdot$  is a twisted intertwining operator of the type*

$$\begin{pmatrix} \phi_{g_1 g_2 g_3}(W_1 \boxtimes_{P(\tilde{z}_0)} W_2) \\ \phi_{g_1 g_2 g_3^2}(W'_4) \ \phi_{g_1 g_2 g_3^{-1}}(W_3) \end{pmatrix}.$$

As a result,  $(\tilde{z}_1, \tilde{z}_2) \in U_1$ ,  $J$  is a  $Q(\tilde{z}_2)$ -intertwining map of type

$$\begin{pmatrix} W_1 \boxtimes_{P(\tilde{z}_0)} W_2 \\ W'_4 W_3 \end{pmatrix}. \quad (10.8)$$

*Proof.* First, we prove the duality property of  $\mathcal{Y}_{B_{\tilde{z}_2}(J)}(\cdot, x)$ . We would study the following three series,

$$\left\langle \eta_2^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1} L(1)} \pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2), \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\tilde{z}_2}(J)}(w'_4, \eta_2) w_3 \right\rangle, \quad (10.9)$$

$$\left\langle \eta_2^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1} L(1)} \pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2), \mathcal{Y}_{B_{\tilde{z}_2}(J)}(w'_4, \eta_2) \phi_{g_1 g_2 g_3^{-1}}(Y)(v, \eta_1) w_3 \right\rangle, \quad (10.10)$$

$$\left\langle \eta_2^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1} L(1)} \pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2), \mathcal{Y}_{B_{\tilde{z}_2}(J)}\left(\phi_{g_1 g_2 g_3^2}(Y)(v, \eta_1 - \eta_2) w'_4, \eta_2\right) w_3 \right\rangle. \quad (10.11)$$

Let  $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{C}^\times$  be complex variables with  $\zeta_0 = \zeta_1 - \zeta_2$ , and  $0 < |\zeta_0| < |\zeta_2| < |\zeta_1|$ . We can define  $J_{\zeta_0, \zeta_2} \in \text{Hom}(W'_4 \otimes W_3, \overline{W_1 \boxtimes_{P(\zeta_0)} W_2})$  to be the linear map uniquely determined by

$$\begin{aligned} J_{\zeta_0, \zeta_2}(w'_4 \otimes w_3)(w_1 \boxtimes_{P(\zeta_0)} w_2) &= \sum_{n \in \mathbb{C}} J_{\zeta_0, \zeta_2}(w'_4 \otimes w_3) (\pi_n(w_1 \boxtimes_{P(\zeta_0)} w_2)) \\ &= \langle w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) w_3 \rangle. \end{aligned} \quad (10.12)$$

Similar as the map  $J$ , the map  $J_{\zeta_0, \zeta_2}$  is well-defined, and we have  $J = J_{\tilde{z}_0, \tilde{z}_2}$ .

To study (10.9)-(10.11), we first study

$$\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1} L(1)} \mathcal{Y}_{\boxtimes_{P(\zeta_0)}}(w_1, \zeta_0) w_2, \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle, \quad (10.13)$$

$$\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) \phi_{g_1 g_2 g_3^{-1}}(Y)(v, \eta_1) w_3 \right\rangle, \quad (10.14)$$

$$\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(\phi_{g_1 g_2 g_3^2}(Y)(v, \eta_1 - \eta_2) w'_4, \eta_2) w_3 \right\rangle, \quad (10.15)$$

where  $\mathcal{Y}_{\boxtimes P(\zeta_0)}(\cdot, x) \cdot$  is the unique twisted intertwining operator such that for any  $w_1 \in W_1, w_2 \in W_2, \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2 = w_1 \boxtimes_{P(\zeta_0)} w_2$ .

First, on the maximal Reinhardt sub-region of the region given by  $0 < \left| \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \right| < |\zeta_2| < |\zeta_1|, 0 < |\eta_1| < |\eta_2|$ , which is the region given by  $0 < 2|\eta_1| < |\eta_2|, 0 < |\zeta_2| < |\zeta_1|$ , (10.16) is absolutely convergent. Therefore, on the region given by  $0 < 2|\eta_1| < |\eta_2|, 0 < |\zeta_0| < |\zeta_2| < |\zeta_1|$ , we know that (10.14) is equal to

$$\begin{aligned} & \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} (w_1 \boxtimes_{P(\zeta_0)} w_2), \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) Y((g_1 g_2 g_3^{-1})^{-1} v, \eta_1) w_3 \right\rangle \\ &= \left\langle w_1 \boxtimes_{P(\zeta_0)} w_2, J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes \right. \right. \\ & \quad \left. \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} Y((g_1 g_2 g_3^{-1})^{-1} v, \eta_1) w_3 \right) \right\rangle \\ &= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) \right. \\ & \quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} Y((g_1 g_2 g_3^{-1})^{-1} v, \eta_1) w_3 \right\rangle \\ &= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) \right. \\ & \quad \left. Y \left( e^{-\zeta_2^{-1}(1-x)L(1)} (1 - \eta_2^{-1} \eta_1)^{-2L(0)} (-\zeta_2 \eta_2^{-1})^{L(0)} (g_1 g_2 g_3^{-1})^{-1} v, \right. \right. \\ & \quad \left. \left. \frac{e^{r\pi i} e^{2l_0(\zeta_2)} e^{l_0(\zeta_2^{-1})} e^{-l_0(\eta_2)} \eta_1}{1-x} \right) e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \Big|_{x=\eta_2^{-1} \eta_1} \end{aligned} \quad (10.16)$$

$$\begin{aligned} &= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) \right. \\ & \quad \left. Y^{p_1} \left( e^{-\zeta_2^{-1}(1-\eta_2^{-1} \eta_1)L(1)} (1 - \eta_2^{-1} \eta_1)^{-2L(0)} (-\zeta_2 \eta_2^{-1})^{L(0)} (g_1 g_2 g_3^{-1})^{-1} v, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \right) \right. \\ & \quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \end{aligned}$$

$$\begin{aligned}
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^{b_1} \left( \zeta_1, \zeta_2, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2}; e^{-\zeta_2^{-1}(1-\eta_2^{-1}\eta_1)L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_2 - \eta_1)^2} \right)^{L(0)} g_3^{-p_1} (g_1 g_2 g_3^{-1})^{-1} v; w_1, w_2, \right. \\
&\quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \quad (10.17)
\end{aligned}$$

where  $p_1 = p_1(r, \zeta_2, \eta_1, \eta_2)$  is the unique integer satisfying

$$l_{p_1} \left( \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \right) = r\pi i + 2l_0(\zeta_2) + l_0(\zeta_2^{-1}) - l_0(\eta_2) + l_0(\eta_1) + \sum_{k>0} \frac{1}{k} \eta_2^{-k} \eta_1^k, \quad (10.18)$$

and  $b_1$  is an element in the fundamental group dependent on the value of  $\zeta_1, \zeta_2$ , and  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2}$ . Since the right-hand side of (10.17) is holomorphic on the region  $0 < \left| \frac{\eta_1}{\eta_2} \right| < 1 < \left| \frac{\zeta_2}{\zeta_0} \right|$ , as a finite sum of series on  $\frac{\eta_1}{\eta_2}$  and  $\frac{\zeta_2}{\zeta_0}$ , we know the left-hand side of (10.17) is absolutely convergent on the region  $0 < \left| \frac{\eta_1}{\eta_2} \right| < 1 < \left| \frac{\zeta_2}{\zeta_0} \right|$ .

We define the region  $\mathcal{R}_1$  by

$$\mathcal{R}_1 = \left\{ (\zeta_0, \zeta_2, \eta_1, \eta_2) \in (\mathbb{C}^\times)^4 \mid \zeta_0 + \zeta_2 \neq 0, \eta_1 \neq \eta_2, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \neq \zeta_1 \right\}, \quad (10.19)$$

equipped with the cuttings  $\zeta_0 \in \mathbb{R}_+, \zeta_2 \in \mathbb{R}_+, \zeta_0 + \zeta_2 \in \mathbb{R}_+, \eta_1 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_+, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_1 \in \mathbb{R}_+, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_2 \in \mathbb{R}_+, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \in \mathbb{R}_+$  (which change the topology and complex structure. All the positive real lines are attached to the upper half complex planes). Because of the cuttings  $\eta_1 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_+, \zeta_2 \in \mathbb{R}_+$ , and  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \in \mathbb{R}_+$ , we know that  $p_1$  is locally constant on

$$\mathcal{R}'_1 = \left\{ (\zeta_0, \zeta_2, \eta_1, \eta_2) \in \mathcal{R}_1 \mid 0 < \left| \frac{\eta_1}{\eta_2} \right| < 1 < \left| \frac{\zeta_2}{\zeta_0} \right| \right\}.$$

Because of the cuttings  $\zeta_i \in \mathbb{R}_+, i = 0, 1, 2, \eta_j \in \mathbb{R}_+, j = 1, 2$ , and  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \in \mathbb{R}_+, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \in \mathbb{R}_+, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_1 \in \mathbb{R}_+$ , and  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_2 \in \mathbb{R}_+$ , we know that  $b_1$  is locally constant on  $\mathcal{R}_1$  and therefore  $\mathcal{R}'_1$ . Because of the cuttings  $\eta_1 \in \mathbb{R}_+$  and  $\eta_2 \in \mathbb{R}_+$ , we know that the function  $\arg \eta_1 - \arg \eta_2$  is continuous on  $\mathcal{R}'_1$ . We pick a point

$q_1 = (-2, -5, -1 - \iota, -5\iota)$  in  $\mathcal{R}'_1$ . On the connected component  $C_1$  of  $\mathcal{R}'_1$  containing  $q_1$ , using  $q_1$  to compute, we know that  $p_1 = 2$ , and  $b_1$  is the unique element such that (10.20) holds. On  $C_1$ , we have

$$\begin{aligned}
& \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) \phi_{g_1 g_2 g_3^{-1}}(Y)(v, \eta_1) w_3 \right\rangle \\
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^{b_1} \left( \zeta_1, \zeta_2, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2}; e^{-\zeta_2^{-1}(1 - \eta_2^{-1} \eta_1)L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_2 - \eta_1)^2} \right)^{L(0)} g_3^{-p_1} (g_1 g_2 g_3^{-1})^{-1} v; \right. \\
&\quad \left. w_1, w_2, e^{-\zeta_2^{-1}L(1)} e^{r\pi \iota L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \\
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2}; e^{-\zeta_2^{-1}(1 - \eta_2^{-1} \eta_1)L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_2 - \eta_1)^2} \right)^{L(0)} (g_1 g_2) g_3^{-2} (g_1 g_2 g_3^{-1})^{-1} v; \right. \\
&\quad \left. w_1, w_2, e^{-\zeta_2^{-1}L(1)} e^{r\pi \iota L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right)
\end{aligned} \tag{10.20}$$

Write  $\eta_0 = \eta_1 - \eta_2$ . On the maximal Reinhardt sub-region of the region given by  $0 < |\zeta_2| < |\zeta_1| < \left| \frac{\eta_2 + \eta_0}{\zeta_2^{-1} \eta_0} \right|$ ,  $0 < |\eta_0| < |\eta_2|$ , which is the region given by  $0 < \left(1 + \frac{|\zeta_1|}{|\zeta_2|}\right) |\eta_0| < |\eta_2|$ ,  $0 < |\zeta_2| < |\zeta_1|$ , (10.21) is absolutely convergent. Therefore, on the region given by  $0 < \left(1 + \frac{|\zeta_1|}{|\zeta_2|}\right) |\eta_0| < |\eta_2|$ ,  $0 < |\zeta_0| < |\zeta_2| < |\zeta_1|$ , we know that (10.15) is equal to

$$\begin{aligned}
& \left\langle e^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} (w_1 \boxtimes_{P(\zeta_0)} w_2), \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(Y((g_1 g_2 g_3^2)^{-1} v, \eta_0) w'_4, \eta_2) w_3 \right\rangle \\
&= \left\langle w_1 \boxtimes_{P(\zeta_0)} w_2, J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} Y((g_1 g_2 g_3^2)^{-1} v, \eta_0) w'_4 \otimes \right. \right. \\
&\quad \left. \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi \iota L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\
&= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} Y((g_1 g_2 g_3^2)^{-1} v, \eta_0) w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) \right. \\
&\quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi \iota L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \\
&= \left\langle Y \left( e^{-\zeta_2(1 + \eta_2^{-1} \eta_0)L(1)} \left( \frac{\eta_2 \zeta_2^{-1}}{(\eta_2 + \eta_0)^2} \right)^{L(0)} (g_1 g_2 g_3^2)^{-1} v, \frac{e^{-l_0(\eta_2)} e^{l_0(\zeta_2^{-1})} \eta_0}{1 + \eta_2^{-1} \eta_0} \right) \right. \\
&\quad \left. e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) \right\rangle
\end{aligned}$$

$$\begin{aligned}
& \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi iL(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \\
&= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \right. \\
& \quad Y \left( e^{\frac{\zeta_2^{-1}\eta_0}{\eta_2+\eta_0}L(1)} \left( -\frac{(\eta_2+\eta_0)^2}{\zeta_2^{-2}\eta_0^2} \right)^{L(0)} e^{-\zeta_2(1+\eta_2^{-1}\eta_0)L(1)} \left( \frac{\eta_2\zeta_2^{-1}}{(\eta_2+\eta_0)^2} \right)^{L(0)} (g_1g_2g_3^2)^{-1}v, \right. \\
& \quad \left. \frac{1+x}{e^{-l_0(\eta_2)}e^{l_0(\zeta_2^{-1})}\eta_0} \right) \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) \\
& \quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi iL(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \Big|_{x=\eta_2^{-1}\eta_0} \\
&= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \right. \\
& \quad Y \left( e^{\frac{\eta_0+\eta_0^2\eta_2^{-1}}{\zeta_2(\eta_2+\eta_0)}L(1)} (-\zeta_2\eta_2\eta_0^{-2})^{L(0)} (g_1g_2g_3^2)^{-1}v, \frac{1+x}{e^{-l_0(\eta_2)}e^{l_0(\zeta_2^{-1})}\eta_0} \right) \\
& \quad \left. \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) e^{-\zeta_2^{-1}L(1)} e^{r\pi iL(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \Big|_{x=\eta_2^{-1}\eta_0} \tag{10.21}
\end{aligned}$$

$$\begin{aligned}
&= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \right. \\
& \quad Y^{p_2} \left( e^{\frac{\eta_0+\eta_0^2\eta_2^{-1}}{\zeta_2(\eta_2+\eta_0)}L(1)} (-\zeta_2\eta_2\eta_0^{-2})^{L(0)} (g_1g_2g_3^2)^{-1}v, \frac{\eta_2+\eta_0}{\zeta_2^{-1}\eta_0} \right) \\
& \quad \left. \mathcal{Y}_1(w_1, \zeta_1) \mathcal{Y}_2(w_2, \zeta_2) e^{-\zeta_2^{-1}L(1)} e^{r\pi iL(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \tag{10.22}
\end{aligned}$$

$$\begin{aligned}
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^{b_2} \left( \zeta_1, \zeta_2, \frac{\eta_2+\eta_0}{\zeta_2^{-1}\eta_0}; e^{\frac{\eta_0+\eta_0^2\eta_2^{-1}}{\zeta_2(\eta_2+\eta_0)}L(1)} (-\zeta_2\eta_2\eta_0^{-2})^{L(0)} (g_1g_2g_3)^{-p_2} (g_1g_2g_3^2)^{-1}v; w_1, w_2, \right. \\
& \quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi iL(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right), \tag{10.23}
\end{aligned}$$

where  $p_2 = p_2(\zeta_2, \eta_0, \eta_2)$  is the unique integer satisfying

$$l_{p_2} \left( \frac{\eta_2+\eta_0}{\zeta_2^{-1}\eta_0} \right) = l_0(\eta_2) - l_0(\zeta_2^{-1}) - l_0(\eta_0) - \sum_{k>0} \frac{(-1)^k}{k} \eta_2^{-k} \eta_0^k, \tag{10.24}$$

and  $b_2$  is an element in the fundamental group. Since the right-hand side of (10.23) is analytic on the region  $0 < \left| \frac{\eta_0}{\eta_2} \right| < 1 < 2 < \left| \frac{\zeta_2}{\zeta_0} \right|$ , we know that the left-hand side of (10.23) is absolutely convergent on this region.

We define the region  $\mathcal{R}_2$  by

$$\mathcal{R}_2 = \left\{ (\zeta_0, \zeta_2, \eta_0, \eta_2) \in (\mathbb{C}^\times)^4 \mid \zeta_0 + \zeta_2 \neq 0, \eta_0 + \eta_2 \neq 0, \frac{\eta_2 + \eta_0}{\zeta_2^{-1}\eta_0} \neq \zeta_1 \right\}, \quad (10.25)$$

equipped with the cuttings  $\zeta_0 \in \mathbb{R}_+, \zeta_2 \in \mathbb{R}_+, \zeta_0 + \zeta_2 \in \mathbb{R}_+, \eta_0 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_+, \frac{\eta_2 + \eta_0}{\zeta_2^{-1}\eta_0} - \zeta_1 \in \mathbb{R}_+, \frac{\eta_2 + \eta_0}{\zeta_2^{-1}\eta_0} - \zeta_2 \in \mathbb{R}_+, \frac{\eta_2 + \eta_0}{\zeta_2^{-1}\eta_0} \in \mathbb{R}_+$  (which changes the topology and complex structure. All the positive real lines are attached to the upper half complex planes). Because of the cuttings  $\eta_0 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_+$ , and  $\frac{\eta_2 + \eta_0}{\zeta_2^{-1}\eta_0} \in \mathbb{R}_+$ , we know that  $p_2$  is locally constant on

$$\mathcal{R}'_2 = \left\{ (\zeta_0, \zeta_2, \eta_0, \eta_2) \in \mathcal{R}_2 \mid 0 < \left| \frac{\eta_0}{\eta_2} \right| < 1 < 2 < \left| \frac{\zeta_2}{\zeta_0} \right| \right\}.$$

Because of all cuttings in  $\mathcal{R}_2$ , we know that  $b_2$  is locally constant on  $\mathcal{R}'_2$ . Because of the cuttings  $\zeta_0 \in \mathbb{R}_+$  and  $\zeta_2 \in \mathbb{R}_+$ , we know that the function  $\arg \zeta_0 - \arg \zeta_2$  is continuous on  $\mathcal{R}_2$  and therefore  $\mathcal{R}'_2$ . We pick a point  $q_2 = (-2, -5, -1 + i, -5i)$  in  $\mathcal{R}'_2$ . On the connected component  $C_2$  of  $\mathcal{R}'_2$  containing  $q_2$ , using  $q_2$  to compute, we know that  $p_2 = -1$ , and  $b_2$  is the identity group element  $e$ . On  $C_2$ , we have

$$\begin{aligned} & \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(\phi_{g_1 g_2 g_3^2}(Y)(v, \eta_1 - \eta_2) w'_4, \eta_2) w_3 \right\rangle \\ &= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\eta_2 + \eta_0}{\zeta_2^{-1}\eta_0}; e^{\frac{\eta_0 + \eta_0^2 \eta_2^{-1}}{\zeta_2(\eta_2 + \eta_0)} L(1)} (-\zeta_2 \eta_2 \eta_0^{-2})^{L(0)} (g_1 g_2 g_3) \cdot (g_1 g_2 g_3^2)^{-1} v; w_1, w_2, \right. \\ & \quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \quad (10.26) \end{aligned}$$

Using  $\eta_0 = \eta_1 - \eta_2$ , there is a natural bijection between  $\mathcal{R}_1$  and  $\mathcal{R}_2$ . Under this identification, the line segment connecting  $q_1$  and  $q_2$  is contained in  $C_1 \cap C_2$ . On  $C_1 \cap C_2$ , by (10.20) and (10.26), we know that

$$\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(\phi_{g_1 g_2 g_3^2}(Y)(v, \eta_1 - \eta_2) w'_4, \eta_2) w_3 \right\rangle$$

$$\begin{aligned}
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\eta_2 + \eta_0}{\zeta_2^{-1} \eta_0}; e^{\frac{\eta_0 + \eta_0^2 \eta_2^{-1}}{\zeta_2 (\eta_2 + \eta_0)} L(1)} (-\zeta_2 \eta_2 \eta_0^{-2})^{L(0)} (g_1 g_2 g_3) \cdot (g_1 g_2 g_3^2)^{-1} v; w_1, w_2, \right. \\
&\quad \left. e^{-\zeta_2^{-1} L(1)} e^{r \pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \\
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2}; e^{-\zeta_2^{-1} (1 - \eta_2^{-1} \eta_1) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_2 - \eta_1)^2} \right)^{L(0)} (g_1 g_2) g_3^{-2} (g_1 g_2 g_3^{-1})^{-1} v; \right. \\
&\quad \left. w_1, w_2, e^{-\zeta_2^{-1} L(1)} e^{r \pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \\
&= \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1} L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) \phi_{g_1 g_2 g_3^{-1}}(Y)(v, \eta_1) w_3 \right\rangle
\end{aligned} \tag{10.27}$$

The right-hand side and the left-hand side are both absolutely convergent series.

Compare the powers of  $\zeta_0$  on the both sides, we have

$$\begin{aligned}
&\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1} L(1)} \pi_n(w_1 \boxtimes_{P(\zeta_0)} w_2), \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(\phi_{g_1 g_2 g_3^2}(Y)(v, \eta_1 - \eta_2) w'_4, \eta_2) w_3 \right\rangle \\
&= \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1} L(1)} \pi_n(w_1 \boxtimes_{P(\zeta_0)} w_2), \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) \phi_{g_1 g_2 g_3^{-1}}(Y)(v, \eta_1) w_3 \right\rangle.
\end{aligned} \tag{10.28}$$

When regard  $\zeta_0, \zeta_2$  as fixed, both sides of (10.28) are absolutely convergent to a single valued branch of a multivalued function on

$$\left\{ (\eta_1, \eta_2) \in (\mathbb{C}^\times)^2 \mid \eta_1 \neq \eta_2, \frac{\eta_2 + \eta_0}{\zeta_2^{-1} \eta_0} \neq \zeta_1, \frac{\eta_2 + \eta_0}{\zeta_2^{-1} \eta_0} \neq \zeta_2, \frac{\eta_2 + \eta_0}{\zeta_2^{-1} \eta_0} \neq 0 \right\}. \tag{10.29}$$

However, when regarding  $\zeta_0$  and  $\zeta_2$  as independent variables, this function is independent on  $\zeta_0$ . After taking a union of (10.29) over some open set of  $\zeta_0$ , we know that the domain of the multivalued function that (10.28) absolutely converges to is

$$\{ (\eta_1, \eta_2) \in (\mathbb{C}^\times)^2 \mid \eta_1 \neq \eta_2 \}. \tag{10.30}$$

It is easy to see that every singularity of this multivalued function is regular.

It can be checked that for any  $(a, b) \in U_1$  (defined in Remark 8.0.7), the set

$$\{(\eta_1, \eta_2) | (a - b, b, \eta_1, \eta_2) \in C_1 \cap C_2\} \quad (10.31)$$

is nonempty open. In particular, for  $(a, b) = (\tilde{z}_1, \tilde{z}_2)$ , we know that (10.10) and (10.11) are absolutely convergent and equal on some nonempty open set, which implies that they are absolutely convergent and equal on the region given by  $0 < |\eta_1| < |\eta_2|$ ,  $0 < |\eta_0| < |\eta_2|$ ,  $\arg(\eta_1 - \eta_2) - \arg(\eta_2) \in (-\frac{3\pi}{2}, -\frac{\pi}{2})$ , and  $|\arg(\eta_1) - \arg(\eta_2)| < \frac{\pi}{2}$ .

Let  $(f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}, f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}^e)$  be a multivalued function with a preferred single valued branch, such that  $f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}$  satisfies Condition 3.0.13 for  $(1, 1, \{0\})$ , and on the region given by  $0 < |\zeta_0| < |\zeta_2|$ , and  $|\arg(\eta_1) - \arg(\eta_2)| < \frac{\pi}{2}$ ,

$$\begin{aligned} & \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \pi_n (w_1 \boxtimes_{P(\zeta_0)} w_2), \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})} \left( \phi_{g_1 g_2 g_3^2}(Y)(v, \eta_1 - \eta_2) w'_4, \eta_2 \right) w_3 \right\rangle \\ &= f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}^e \left( \eta_1, \eta_2; v, w'_4, w_3; \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \pi_n (w_1 \boxtimes_{P(\zeta_0)} w_2) \right). \end{aligned} \quad (10.32)$$

We know that there uniquely exists such a pair  $(f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}, f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}^e)$  because of the part (e) of Condition 3.0.13 applying to coefficients of the expansion of

$$f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), 3} \left( \zeta_0, -\zeta_2, \frac{\zeta_2 \eta_2}{\eta_1 - \eta_2} \right)$$

at the regular singularity  $\zeta_0 = 0$ .

On the region given by  $(\zeta_0, \zeta_2, \eta_1, \eta_2) \in C_2$  and  $|\arg(\eta_1) - \arg(\eta_2)| < \frac{\pi}{2}$ , by (10.26) and (10.32), we have

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\eta_2 + \eta_0}{\zeta_2^{-1} \eta_0}; e^{\frac{\eta_0 + \eta_0^2 \eta_2^{-1}}{\zeta_2^2 (\eta_2 + \eta_0)} L(1)} (-\zeta_2 \eta_2 \eta_0^{-2})^{L(0)} (g_1 g_2 g_3) \cdot (g_1 g_2 g_3^2)^{-1} v; w_1, w_2, \right. \\ & \quad \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \\ &= \sum_{n \in \mathbb{C}} \sum_{k \in \mathbb{N}} f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2}), 2}}^e \left( \eta_1, \eta_2; v, w'_4, w_3; \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} (\mathcal{Y}_{\boxtimes_{P(\zeta_0)}})_{-n-1, k}(w_1) w_2 \right) \end{aligned}$$

$$\zeta_0^{n+\text{wt } w_1+\text{wt } w_2} \log(\zeta_0)^k. \quad (10.33)$$

We define the region  $\mathcal{R}_0$  by

$$\mathcal{R}_0 = \left\{ (\zeta_0, \zeta_2, \eta_1, \eta_2) \in (\mathbb{C}^\times)^4 \mid \zeta_0 + \zeta_2 \neq 0, \eta_1 \neq \eta_2, \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \neq \zeta_1 \right\}, \quad (10.34)$$

equipped with the cuttings  $\zeta_0 \in \mathbb{R}_+$ ,  $\zeta_2 \in \mathbb{R}_+$ ,  $\zeta_0 + \zeta_2 \in \mathbb{R}_+$ ,  $\eta_1 \in \mathbb{R}_+$ ,  $\eta_2 \in \mathbb{R}_+$ ,  $\eta_1 - \eta_2 \in \mathbb{R}_+$ ,  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_1 \in \mathbb{R}_+$ ,  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_2 \in \mathbb{R}_+$ ,  $\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} \in \mathbb{R}_+$  (which change the topology and complex structure. All the positive real lines are attached to the upper half complex planes). Define

$$\mathcal{R}'_0 = \left\{ (\zeta_0, \zeta_2, \eta_1, \eta_2) \in \mathcal{R}_0 \mid |\zeta_0| < \min\{|\zeta_2|, |\eta_2 \zeta_2 / \eta_0|\} \right\}. \quad (10.35)$$

Because of these cuttings, the left-hand side of (10.33) is single-valued analytic on  $\mathcal{R}_0$ , and the both sides of (10.33) are single-valued analytic on  $\mathcal{R}'_0$ . Therefore, we know that (10.33) holds on the union  $C_0$  of connected components of  $\mathcal{R}'_0$  whose intersection with the region given by  $(\zeta_0, \zeta_2, \eta_1, \eta_2) \in C_2$  and  $|\arg(\eta_1) - \arg(\eta_2)| < \frac{\pi}{2}$  have nonempty interior.

Now, we study (10.13). On the region given by  $0 < |\zeta_2| + \left| \frac{\zeta_2 \eta_2}{\eta_1 - \eta_2} - \zeta_0 \right| < |\zeta_1|$ , we have

$$\begin{aligned} & \left\langle \mathcal{Y}_{\boxtimes P(\zeta_0)} \left( Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right) w_1, \zeta_0 \right) w_2, \right. \\ & \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\ &= \left\langle e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4, \right. \\ & \quad \mathcal{Y}_1 \left( Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right) w_1, \zeta_1 \right) \\ & \quad \left. \mathcal{Y}_2(w_2, \zeta_2) e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right\rangle \quad (10.36) \end{aligned}$$

$$\begin{aligned}
&= f_{\mathcal{Y}_{1, \mathcal{Y}_{2, 3}}^{b_3}} \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v; w_1, w_2, \right. \\
&\quad \left. e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right) \quad (10.37)
\end{aligned}$$

The expression (10.36) is absolute convergent on the region given by  $0 < |\zeta_2| + \left| \frac{\zeta_2 \eta_2}{\eta_1 - \eta_2} - \zeta_0 \right| < |\zeta_1|$ , to a single-valued branch (10.37) of its multivalued function. Let

$$\mathcal{R}'_3 = \left\{ (\zeta_0, \zeta_2, \eta_1, \eta_2) \in \mathcal{R}_1 \mid 0 < |\zeta_2| + \left| \frac{\zeta_2 \eta_2}{\eta_1 - \eta_2} - \zeta_0 \right| < |\zeta_1| \right\}.$$

Then,  $b_3$  is locally constant on  $\mathcal{R}'_3$ , and (10.37) is single valued analytic on  $\mathcal{R}'_3$ . Let  $q_3 = (-2, -14, -25i, -5i) \in \mathcal{R}'_3$ . We have  $(\zeta_1, \zeta_2, \frac{\zeta_2 \eta_2}{\eta_1 - \eta_2} - \zeta_0 + \zeta_1) = (-16, -14, -17.5)$  at  $q_3$ . Using Proposition 8.0.4, we know that  $b_3 = e$  on the connected component  $C_3$  of  $\mathcal{R}'_3$  containing  $q_3$ . Therefore, on  $C_3$  we have

$$\begin{aligned}
&\left\langle \mathcal{Y}_{\boxtimes P(\zeta_0)} \left( Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right) w_1, \zeta_0 \right) w_2, \right. \\
&\quad \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\
&= f_{\mathcal{Y}_{1, \mathcal{Y}_{2, 3}}^e} \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v; w_1, w_2, \right. \\
&\quad \left. e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right). \quad (10.38)
\end{aligned}$$

Also, on the region given by

$$\begin{aligned}
0 < \left| \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right| < |\zeta_0| < \left| \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right|, \\
\left| \arg \left( \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) - \arg(\zeta_0) \right| < \frac{\pi}{2}, \quad \left| \arg \left( \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right) - \arg(\zeta_0) \right| < \frac{\pi}{2}, \quad (10.39)
\end{aligned}$$

we know that as elements in  $\overline{W_1 \boxtimes_{P(\zeta_0)} W_2}$ ,

$$\begin{aligned} & \mathcal{Y}_{\boxtimes_{P(\zeta_0)}} \left( Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right) w_1, \zeta_0 \right) w_2 \\ &= Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) \mathcal{Y}_{\boxtimes_{P(\zeta_0)}} (w_1, \zeta_0) w_2. \end{aligned} \quad (10.40)$$

Therefore, on

$$\{(\zeta_0, \zeta_2, \eta_1, \eta_2) \in C_3 \mid \text{Conditions (10.39) hold.}\}, \quad (10.41)$$

we have

$$\begin{aligned} & \left\langle Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) \mathcal{Y}_{\boxtimes_{P(\zeta_0)}} (w_1, \zeta_0) w_2, \right. \\ & \quad \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\ &= \left\langle \mathcal{Y}_{\boxtimes_{P(\zeta_0)}} \left( Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} - \zeta_0 \right) w_1, \zeta_0 \right) w_2, \right. \\ & \quad \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\ &= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v; w_1, w_2, \right. \\ & \quad \left. e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right). \end{aligned} \quad (10.42)$$

The region where the left-hand side of (10.42) is equal to the right-hand side of (10.42) (i.e., ignore the intermediate term) can be enlarged from (10.41) to a larger region as the following. Notice that the left-hand side of (10.42) is a *finite* linear combination (whose coefficients are of the form  $\eta_2^a \zeta_2^b (\eta_1 - \eta_2)^c \log(\eta_2)^d \log(\zeta_2)^e$ ,  $a, b \in \mathbb{C}$ ,  $c \in \mathbb{Z}$ ,  $d, e \in \mathbb{N}$ ) of series in  $\frac{\zeta_2 \eta_2}{\zeta_1 - \zeta_2}$ , and  $\zeta_0$ , whose coefficients are of the form  $\langle a_1, J_{\zeta_0, \zeta_2}(a_2 \otimes a_3) \rangle$  ( $a_1, a_2, a_3$  are module elements instead of elements in algebraic completion of

modules), which can be written as finite linear combinations of monomials of the form  $\zeta_2^a \zeta_0^b \log(\zeta_2)^c \log(\zeta_0)^d$ ,  $a, b \in \mathbb{C}$ ,  $c, d \in \mathbb{Z}$ . Therefore, the left-hand side of (10.42) is an iterate series whose inner sum (i.e. coefficient) are in  $\zeta_2$  and the outer sum is in  $\left(\frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_2\right)$  and  $\zeta_0$ . We already know (10.42) is absolutely convergent on the region (10.41). We know that there is an expansion (multiple series) in  $\zeta_0$ ,  $X - \zeta_2$ , and  $\zeta_2$  of  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(\zeta_1, \zeta_2, X)$  which absolutely converges on the region given by  $0 < |\zeta_0| < |X - \zeta_2| < |\zeta_2|$ . This is because  $\left(\frac{\zeta_1 - \zeta_2}{X - \zeta_2}, \frac{\zeta_2}{X - \zeta_2}\right) = (0, \infty)$  is a regular singularity of the function  $f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), 3}\left(\frac{\zeta_1 - \zeta_2}{X - \zeta_2}, \frac{-\zeta_2}{X - \zeta_2}, 1\right)$ , which has relation with  $f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), 3}(\zeta_1 - \zeta_2, -\zeta_2, X - \zeta_2)$  (using  $L(0)$ -conjugation property) and therefore  $f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}(\zeta_1, \zeta_2, X)$ . Let  $X = \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2}$ . Since the intersection of the region given by  $0 < |\zeta_0| < |X - \zeta_2| < |\zeta_2|$  and the region (10.41) is open and nonempty, using unique expansion property, we know that they are the same series. Therefore, we know that as a series in  $\zeta_0$ ,  $X - \zeta_2$  and  $\zeta_2$ , the left-hand side of (10.42) is absolutely convergent on the region given by

$$0 < |\zeta_0| < \left| \frac{\zeta_2 \eta_1}{\eta_1 - \eta_2} - \zeta_2 \right| < |\zeta_2|. \quad (10.43)$$

Moreover, on the region given by (10.43), and the second line of (10.39),  $(\zeta_0, \zeta_2, \eta_1, \eta_2) \in C_3$  we have

$$\begin{aligned} & \left\langle Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) \mathcal{Y}_{\mathbb{R}P(\zeta_0)}(w_1, \zeta_0) w_2, \right. \\ & \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\ & = f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^b \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} v; w_1, w_2, \right. \\ & \left. e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right), \quad (10.44) \end{aligned}$$

for some group element  $b$ . Therefore, since we have

$$\begin{aligned}
& \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle \\
&= \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, Y((g_1 g_2 g_3)^{-1} v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle \\
&= \left\langle e^{\zeta_2^{-1}L(1)} (\zeta_2^{-1})^{-L(0)} \eta_2^{L(0)} Y^o((g_1 g_2 g_3)^{-1} v, \eta_1) \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \right. \\
&\quad \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\
&= \left\langle Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2 g_3)^{-1} v, \frac{\eta_2 e^{-l_0(\eta_1) - l_0(\zeta_2^{-1})}}{1-x} \right) \right. \\
&\quad \left. \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes \right. \right. \\
&\quad \left. \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \Big|_{x=\eta_2 \eta_1^{-1}}, \tag{10.45}
\end{aligned}$$

we know, by taking the maximal Reinhardt sub-region of (10.43), that the left-hand side of (10.45), which is the same as (10.13), is absolutely convergent on the region given by

$$2 < \left| \frac{\eta_1}{\eta_2} \right|, \quad 0 < \left| \frac{\eta_1}{\eta_2} \right| + 1 < \left| \frac{\zeta_2}{\zeta_0} \right|. \tag{10.46}$$

On the region given by (10.46) and the second line of (10.39), we have

$$\begin{aligned}
& \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle \\
&= \left\langle Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2 g_3)^{-1} v, \frac{\eta_2 e^{-l_0(\eta_1) - l_0(\zeta_2^{-1})}}{1-x} \right) \right. \\
&\quad \left. \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes \right. \right. \\
&\quad \left. \left. e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \Big|_{x=\eta_2 \eta_1^{-1}} \\
&= \left\langle Y^{p_3} \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2 g_3)^{-1} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \right.
\end{aligned}$$

$$\begin{aligned}
& \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\
&= \left\langle Y \left( e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2)^{-p_3} (g_1 g_2 g_3)^{-1} v, \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) \right. \\
&\quad \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2 \left. J_{\zeta_0, \zeta_2} \left( e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \otimes \right. \right. \\
&\quad \left. \left. e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3 \right) \right\rangle \\
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^{b_4} \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2)^{-p_3} (g_1 g_2 g_3)^{-1} v; \right. \\
&\quad \left. w_1, w_2, e^{-\zeta_2^{-1} L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right), \tag{10.47}
\end{aligned}$$

where  $b_4$  is an element in the fundamental group and  $p_3 = p_3(\zeta_2, \eta_1, \eta_2)$  is the unique integer satisfying

$$l_{p_3} \left( \frac{\eta_2 \zeta_2}{\eta_1 - \eta_2} \right) = l_0(\eta_2) - l_0(\eta_1) - l_0(\zeta_2^{-1}) + \sum_{k>0} \frac{1}{k} \eta_2^k \eta_1^{-k}. \tag{10.48}$$

We define the region

$$\mathcal{R}'_4 = \{(\zeta_0, \zeta_2, \eta_1, \eta_2) \in \mathcal{R}_1 \mid 2 < |\eta_1/\eta_2|, \quad 0 < |\eta_1/\eta_2| + 1 < |\zeta_2/\zeta_0|\} \tag{10.49}$$

Notice that  $q_3 \in \mathcal{R}'_4$ , and also satisfies the second line of the condition (10.39). Let  $C_4$  be the connected component of  $\mathcal{R}'_4$  containing  $q_3$ . Because of those cuttings on  $\mathcal{R}_1$ ,  $p_3$  and  $b_4$  are both locally constant on  $\mathcal{R}_1$  and therefore  $\mathcal{R}'_4$ . Therefore,  $p_3$  and  $b_4$  are constant on  $C_4$ . Using  $q_3$  to compute, we obtain that  $p_3 = -1$  on  $C_4$ . Using (10.44) and the fact that  $q_3$  is contained in  $C_3$  and also satisfies the second line of the condition (10.39), we know that  $b_4 = e$  on  $C_4$ . Therefore on  $C_4$ , we have

$$\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1} L(1)} \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle$$

$$\begin{aligned}
&= f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2) (g_1 g_2 g_3)^{-1} v; \right. \\
&\quad \left. w_1, w_2, e^{-\zeta_2^{-1} L(1)} e^{r \pi L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_4' \right)
\end{aligned} \tag{10.50}$$

Therefore, for any  $n \in \mathbb{C}$ ,

$$\left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1} L(1)} \pi_n \mathcal{Y}_{\boxtimes P(\zeta_0)}(w_1, \zeta_0) w_2, \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w_4', \eta_2) w_3 \right\rangle \tag{10.51}$$

is a coefficient of the expansion of the right-hand side of (10.50) at the regular singularity  $\zeta_0 = 0$ . By the part (e) of Condition 3.0.13, we know that (10.51), as a function in  $-\zeta_2$  and  $\frac{\zeta_2 \eta_2}{\eta_1 - \eta_2} = \frac{\zeta_2 \cdot (\eta_2 \eta_1^{-1})}{1 - (\eta_2 \eta_1^{-1})}$  is absolutely convergent, on the region given by  $2 < |\eta_1 / \eta_2|$ , to a multivalued function satisfies Condition 3.0.13 for  $(1, 1, \{0\})$ . We know that the expansion at  $(X_1, X_2) = (\infty, 0)$  of a multivalued function in  $X_1$  and  $X_2$  satisfying Condition 3.0.13 for  $(1, 1, \{0\})$  would have the region of convergence  $0 < |X_2| < |X_1|$ . Therefore, The region of convergence of (10.51) can be enlarged to  $r < |\eta_1 / \eta_2|$ , for any  $r \geq 1$  satisfying the condition

$$-\zeta_2, 0 \notin \left\{ \frac{\zeta_2 \cdot (\eta_2 \eta_1^{-1})}{1 - (\eta_2 \eta_1^{-1})} \mid |\eta_1 / \eta_2| > r \right\}$$

We know that  $r = 1$  satisfies the above condition. Therefore, on the region

$$\mathcal{R}'_5 = \{(\zeta_0, \zeta_2, \eta_1, \eta_2) \in \mathcal{R}_1 \mid |\eta_1 / \eta_2| > 1\}, \tag{10.52}$$

the series (10.51) is absolutely convergent.

On  $C_4$ , by (10.50), we know that

$$f_{\mathcal{Y}_1, \mathcal{Y}_2, 3}^e \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_2}; e^{\zeta_2^{-1} \eta_2^{-1} (\eta_1 - \eta_2) L(1)} \left( \frac{-\zeta_2 \eta_2}{(\eta_1 - \eta_2)^2} \right)^{L(0)} (g_1 g_2) (g_1 g_2 g_3)^{-1} v; \right.$$

$$\begin{aligned}
& w_1, w_2, e^{-\zeta_2^{-1}L(1)} e^{r\pi i L(0)} \zeta_2^{2L(0)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \rangle \\
&= \sum_{n \in \mathbb{C}, k \in \mathbb{N}} \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} (\mathcal{Y}_{\boxtimes P(\zeta_0)})_{-n-1, k}(w_1)(w_2), \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \right. \\
&\quad \left. \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle \zeta_0^{n+\text{wt } w_1 + \text{wt } w_2} \log(\zeta_0)^k. \tag{10.53}
\end{aligned}$$

Equipped the coarsest topology on  $C_4 \cap C_0$  such that both embeddings  $C_4 \cap C_0 \hookrightarrow \mathcal{R}_1$  and  $C_4 \cap C_0 \hookrightarrow \mathcal{R}_0$  are continuous, which is equivalent to the subspace topology induced by  $C_4 \cap C_0 \subset \mathcal{R}_0$ . It can be check that  $q_2$  and  $q_3$  are both contained in  $\mathcal{R}'_0$ , and can be connected by a continuous path in  $\mathcal{R}'_0$ . Therefore, since  $q_2 \in C_0$ , we know that  $q_3 \in C_0$ , and  $C_4 \cap C_0$  has nonempty interior.

On the region given by  $C_4 \cap C_0$ , compare the coefficient of  $\zeta_0^{n+\text{wt } w_1 + \text{wt } w_2} \log(\zeta_0)^k$ , in (10.33) and (10.53), we know that

$$\begin{aligned}
& \left\langle \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} (\mathcal{Y}_{\boxtimes P(\zeta_0)})_{-n-1, k}(w_1)(w_2), \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}(w'_4, \eta_2) w_3 \right\rangle \\
&= f_{\mathcal{Y}_{B_{\zeta_2}(J_{\zeta_0, \zeta_2})}, 2}^e \left( \eta_1, \eta_2; v, w'_4, w_3; \eta_2^{-L(0)} (\zeta_2^{-1})^{L(0)} e^{-\zeta_2^{-1}L(1)} (\mathcal{Y}_{\boxtimes P(\zeta_0)})_{-n-1, k}(w_1) w_2 \right) \\
&\tag{10.54}
\end{aligned}$$

Both sides of (10.54) is analytic and single-valued on the region given by  $0 < |\eta_2| < |\eta_1|$ , and  $|\arg(\eta_1 - \eta_2) - \arg(\eta_1)| < \frac{\pi}{2}$ , they are equal in a nonempty open subset  $C_4 \cap C_0$ , we know that they are equal on the region given by  $0 < |\eta_2| < |\eta_1|$ , and  $|\arg(\eta_1 - \eta_2) - \arg(\eta_1)| < \frac{\pi}{2}$ . It can be checked that for any  $(\zeta_1, \zeta_2) \in U_1$  (see the definition of  $U_1$  in Remark 8.0.7), the set  $\{(\eta_1, \eta_2) \in F_2(\mathbb{C}^\times) \mid (\zeta_1 - \zeta_2, \zeta_2 \eta_1, \eta_2) \in C_4 \cap C_0\}$  is always a subset of region given by  $0 < |\eta_2| < |\eta_1|$ , and  $|\arg(\eta_1 - \eta_2) - \arg(\eta_1)| < \frac{\pi}{2}$  having nonempty interior. Therefore, in particular for  $(\zeta_1, \zeta_2) = (\tilde{z}_1, \tilde{z}_2)$ , we know that

$$\left\langle \eta_2^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1}L(1)} \pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2), \phi_{g_1 g_2 g_3}(Y)(v, \eta_1) \mathcal{Y}_{B_{\tilde{z}_2}(J)}(w'_4, \eta_2) w_3 \right\rangle$$

$$= f_{\mathcal{Y}_{B_{\tilde{z}_2}(J),2}}^e \left( \eta_1, \eta_2; v, w'_4, w_3; \eta_2^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1}L(1)} (\mathcal{Y}_{\boxtimes_{P(\tilde{z}_0)}})_{-n-1,k}(w_1)w_2 \right). \quad (10.55)$$

So far, the duality property for  $\mathcal{Y}_{B_{\tilde{z}_2}(J)}$  is proved.

The final step is to prove that

$$\begin{aligned} & \left\langle \eta_{k+1}^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1}L(1)} \pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2) \right. \\ & \left. \phi_{g_1 g_2 g_3}(Y)(v_1, \eta_1) \cdots \phi_{g_1 g_2 g_3}(Y)(v_k, \eta_k) \mathcal{Y}_{B_{\tilde{z}_2}(J)}(w'_4, \eta_{k+1}) w_3 \right\rangle, \end{aligned} \quad (10.56)$$

is absolutely convergent, on the region given by  $0 < |\eta_{k+1}| < \cdots < |\eta_1|$ , to a single-valued branch of a multivalued function satisfying Condition 3.0.13 for  $(k, 1, \{0\})$ . Using a similar argument as the  $k = 1$  case we have already studied above, we can show that (10.56) is absolutely convergent, on the region given by  $0 < |\eta_{k+1}| < \cdots < |\eta_1|$ , to a multivalued function, which we denote by

$$f_{\mathcal{Y}_{B_{\tilde{z}_2}(J),k+1}}^e(\eta_1, \dots, \eta_{k+1}; v_1, \dots, v_k, w'_4, w_3, \eta_{k+1}^{-L(0)} (\tilde{z}_2^{-1})^{L(0)} e^{-\tilde{z}_2^{-1}L(1)} \pi_n(w_1 \boxtimes_{P(\tilde{z}_0)} w_2)). \quad (10.57)$$

Moreover, we can show that (10.57) converges to the coefficient of the series expansion of (10.58) at the regular singularity  $\zeta_0 = 0$ . Here, we view (10.58) as a function in  $\zeta_0$ ,  $-\zeta_2$ , and  $\frac{\eta_{k+1}\zeta_2}{\eta_1 - \eta_{k+1}}, \dots, \frac{\eta_{k+1}\zeta_2}{\eta_k - \eta_{k+1}}$ . One can consider  $f_{\mathcal{Y}_1, \Omega_+(\mathcal{Y}_2), k+2}$  if that helps understand.

$$\begin{aligned} & f_{\mathcal{Y}_1, \mathcal{Y}_2, k+2} \left( \zeta_1, \zeta_2, \frac{\eta_1 \zeta_2}{\eta_1 - \eta_{k+1}}, \dots, \frac{\eta_k \zeta_2}{\eta_k - \eta_{k+1}}; \right. \\ & e^{\zeta_2^{-1} \eta_{k+1}^{-1} (\eta_1 - \eta_{k+1}) L(1)} \left( \frac{-\zeta_2 \eta_{k+1}}{(\eta_1 - \eta_{k+1})^2} \right)^{L(0)} (g_1 g_2) (g_1 g_2 g_3)^{-1} v_1, \dots \\ & e^{\zeta_2^{-1} \eta_{k+1}^{-1} (\eta_k - \eta_{k+1}) L(1)} \left( \frac{-\zeta_2 \eta_{k+1}}{(\eta_k - \eta_{k+1})^2} \right)^{L(0)} (g_1 g_2) (g_1 g_2 g_3)^{-1} v_k, w_1, w_2, \\ & \left. e^{-\zeta_2^{-1} L(1)} e^{r \pi i L(0)} \zeta_2^{2L(0)} \eta_{k+1}^{-L(0)} (\zeta_2^{-1})^{L(0)} w_3, e^{-\zeta_2 L(1)} \eta_{k+1}^{-L(0)} (\zeta_2^{-1})^{L(0)} w'_4 \right). \end{aligned} \quad (10.58)$$

Since (10.58) satisfies Condition 3.0.13 for  $(k, 2, \{0\})$ , using the part (d) of Condition 3.0.13, we know that (10.57) satisfies Condition 3.0.13 for  $(k, 1, \{0\})$ .

□

# Chapter 11

## Associativities

### 11.1 Associativity of twisted intertwining operators

We say that the category  $\mathcal{C}$  satisfies the **associativity of twisted intertwining operators**, if the following two properties are satisfied.

1. Let  $W_1, W_2, W_3, M_1, W_4$  be any  $g_1$ -,  $g_2$ -,  $g_3$ -,  $g_2g_3$ -,  $g_1g_2g_3$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. For any twisted intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of types  $\binom{W_4}{W_1 M_1}$ ,  $\binom{M_1}{W_2 W_3}$ , respectively, there exists a  $g_1g_2$ -twisted  $V$ -module  $M_2$  in  $\mathcal{C}$ , and twisted intertwining operators  $\mathcal{Y}_3, \mathcal{Y}_4$  of types  $\binom{W_4}{M_2 W_3}$ ,  $\binom{M_2}{W_1 W_2}$ , respectively, such that for any  $w_i \in W_i$ ,  $i = 1, 2, 3$ , and  $w'_4 \in W'_4$ , the equation (11.2) holds on the region given by

$$\begin{aligned} 0 < |z_1 - z_2| < |z_2| < |z_1|, \\ |\arg(z_1 - z_2) - \arg(z_2)| < \frac{\pi}{2}, |\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}. \end{aligned} \quad (11.1)$$

2. Let  $W_1, W_2, W_3, M_2, W_4$  be any  $g_1$ -,  $g_2$ -,  $g_3$ -,  $g_1g_2$ -,  $g_1g_2g_3$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. For any twisted intertwining operators  $\mathcal{Y}_3, \mathcal{Y}_4$  of types  $\binom{W_4}{M_2 W_3}$ ,

$\binom{M_2}{W_1 \ W_2}$ ), respectively, there exists a  $g_2g_3$ -twisted  $V$ -module  $M_1$  in  $\mathcal{C}$ , and twisted intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of types  $\binom{W_4}{W_1 \ M_1}, \binom{M_1}{W_2 \ W_3}$ , respectively, such that for any  $w_i \in W_i, i = 1, 2, 3$ , and  $w'_4 \in W'_4$ , the equation (11.2) holds on the region given by (11.1).

$$\langle w'_4, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3 \rangle = \langle w'_4, \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle \quad (11.2)$$

In this section, we shall prove the associativity of twisted intertwining operators for a **nice enough** category  $\mathcal{C}$ .

**Theorem 11.1.1.** *Suppose  $\mathcal{C}$  satisfies assumptions 6.0.7, 8.0.1, and 8.0.5. In other words,*

1. *For any twisted module  $W$  in  $\mathcal{C}$ , the nilpotent part  $L_W(0)_N$  of  $L_W(0)$  is nilpotent on  $W$ .*
2. *For any twisted module  $W$  in  $\mathcal{C}$ ,  $L_W(1)$  is locally nilpotent.*
3. *For any  $z \in \mathbb{C}^\times$ , and twisted modules  $W_1$  and  $W_2$  in  $\mathcal{C}$ ,  $W_1 \boxtimes_{P(z)} W_2$  is also in  $\mathcal{C}$*
4. *The contragredient of an object in  $\mathcal{C}$  is also in  $\mathcal{C}$ . The direct sum of two objects in  $\mathcal{C}$  is also in  $\mathcal{C}$ .*
5. *For any twisted module  $W$  in  $\mathcal{C}$  and  $n \in \mathbb{C}$ ,  $\dim(W_{[n]}) < \infty$ . In particular,  $W \cong W''$ .*
6.  *$\mathcal{C}$  is closed under taking image.*
7. *Let  $W_1, W_2, W_3, M_1, M_2, W_4$  be any  $g_1^-$ ,  $g_2^-$ ,  $g_3^-$ ,  $g_2g_3^-$ ,  $g_1g_2^-$ ,  $g_1g_2g_3^-$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. Let  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$  be any twisted intertwining operators of types  $\binom{W_4}{W_1 \ M_1}, \binom{M_1}{W_2 \ W_3}, \binom{W_4}{M_2 \ W_3}, \binom{M_2}{W_1 \ W_2}$ , respectively.*

(a)  $(\mathcal{Y}_1, \mathcal{Y}_2)$  satisfies the regular singular point property and the convergence and extension property for products.

(b)  $(\mathcal{Y}_3, \mathcal{Y}_4)$  satisfies the regular singular point property and the convergence and extension property for iterates.

Furthermore, we assume

8. Let  $W_1, W_2, W_3, M_1, \mathcal{Y}_1, \mathcal{Y}_2$  be as in 7. For any  $w_3 \in W_3, w'_4 \in W'_4, z_1, z_2 \in \mathbb{C}$  satisfying (11.1), and  $n \in \mathbb{C}$ , the functional  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) \in (W_1 \otimes W_2)^*$  defined in (9.4) is  $P(z_1 - z_2)$ - $\mathcal{C}$ -embeddable.

Then, the associativity of twisted intertwining operators for  $\mathcal{C}$  holds.

*Proof.* We first prove the part 1. of associativity of twisted intertwining operators.

Let  $W_1, W_2, W_3, M_1, W_4$  be any  $g_1$ -,  $g_2$ -,  $g_3$ -,  $g_2g_3$ -,  $g_1g_2g_3$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. Let  $\mathcal{Y}_1, \mathcal{Y}_2$  be twisted intertwining operators of types  $\binom{W_4}{W_1 M_1}, \binom{M_1}{W_2 W_3}$ , respectively. Let  $z_1, z_2 \in \mathbb{C}$  satisfying (11.1).

For any  $n \in \mathbb{C}$ , recall that in (9.4) we have defined  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) \in (W_1 \otimes W_2)^*$ . By Theorem 9.0.1, we know that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2)$  is  $P(z_1 - z_2)$ -compatible for any  $w_3 \in W_3$  and  $w'_4 \in W'_4$ . By the fact that  $\mathcal{C}$  is closed under taking image, and  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2)$  is  $P(z_1 - z_2)$ - $\mathcal{C}$ -embeddable, applying Theorem 7.0.23, we know that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) \in W_1 \boxtimes_{P(z_1 - z_2)} W_2$ . Also, we know that  $\text{wt} \left( \tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) \right) = n$ . Let  $\Lambda(w_3, w'_4; z_1, z_2)$  be the element in  $\overline{W_1 \boxtimes_{P(z_1 - z_2)} W_2}$  such that

$$\pi_n \left( \Lambda(w_3, w'_4; z_1, z_2) \right) = \tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2), \quad \text{for any } n \in \mathbb{C}.$$

We naturally identify the spaces  $\overline{W_1 \boxtimes_{P(z_1 - z_2)} W_2}$  and  $(W_1 \boxtimes_{P(z_1 - z_2)} W_2)^*$ . Therefore, we can view  $\Lambda(w_3, w'_4; z_1, z_2)$  as a linear functional in  $(W_1 \boxtimes_{P(z_1 - z_2)} W_2)^*$ . We have

$$\Lambda(w_3, w'_4; z_1, z_2) \left( w_1 \boxtimes_{P(z_1 - z_2)} w_2 \right)$$

$$\begin{aligned}
&= \sum_{n \in \mathbb{C}} \tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) (\pi_n w_1 \boxtimes_{P(z_1-z_2)} w_2) \\
&= \langle w'_4, \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_3, z_2) w_3 \rangle.
\end{aligned} \tag{11.3}$$

The last step is because of the definition of  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2)$ . It is easy to check that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2)$ ,  $n \in \mathbb{C}$ , and therefore  $\Lambda(w_3, w'_4; z_1, z_2)$  are bilinear in  $w_3$  and  $w'_4$ .

Consider the map

$$\begin{aligned}
W'_4 \otimes W_3 &\rightarrow \overline{W_1 \boxtimes_{P(z_1-z_2)} W_2} \\
w'_4 \otimes w_3 &\mapsto \Lambda(w_3, w'_4; z_1, z_2)
\end{aligned} \tag{11.4}$$

Recall (10.4), the map (11.4) is equal to  $J$  with  $\tilde{z}_1 = z_1$  and  $\tilde{z}_2 = z_2$ . Since  $z_1, z_2$  satisfies (11.1), by Theorem 10.0.1, we know that the map (11.4) is a  $Q(z_2)$ -intertwining map of type  $\begin{pmatrix} W_1 \boxtimes_{P(z_0)} W_2 \\ W'_4 \ W_3 \end{pmatrix}$ . In other words, There exists a twisted intertwining operator  $\mathcal{Y}_4$  of type  $\begin{pmatrix} W_4 \\ W_1 \boxtimes_{P(z_1-z_2)} W_2 \ W_3 \end{pmatrix}$ , such that for any  $w \in W_1 \boxtimes_{P(z_1-z_2)} W_2$

$$\Lambda(w_3, w'_4; z_1, z_2)w = \langle w'_4, \mathcal{Y}_4(w, z_1 - z_2)w_3 \rangle. \tag{11.5}$$

Therefore, for any  $w_1 \in W_1$ ,  $w_2 \in W_2$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , we have

$$\begin{aligned}
&\langle w'_4, \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_2, z_2) w_3 \rangle \\
&= \Lambda(w_3, w'_4; z_1, z_2) (w_1 \boxtimes_{P(z_1-z_2)} w_2) \\
&= \sum_{n \in \mathbb{C}} \Lambda(w_3, w'_4; z_1, z_2) (\pi_n w_1 \boxtimes_{P(z_1-z_2)} w_2) \\
&= \sum_{n \in \mathbb{C}} \langle w'_4, \mathcal{Y}_4(\pi_n w_1 \boxtimes_{P(z_1-z_2)} w_2, z_1 - z_2) w_3 \rangle \\
&\stackrel{(*)}{=} \langle w'_4, \mathcal{Y}_4(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_1 - z_2) w_3 \rangle \\
&= \langle w'_4, \mathcal{Y}_4(\mathcal{Y}_{\boxtimes_{P(z_1-z_2)}}(w_1, z_1 - z_2)w_2, z_1 - z_2)w_3 \rangle
\end{aligned}$$

Notices that the  $(*)$  step is because  $W_1 \boxtimes_{P(z_1-z_2)} W_2$  is an object in  $\mathcal{C}$ , and  $\langle w'_4, \mathcal{Y}_4(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_1 - z_2) w_3 \rangle$  is absolutely convergent. Let  $\mathcal{Y}_3$  be  $\mathcal{Y}_{\boxtimes_{P(z_1-z_2)}}$ .

Using operator  $\Omega_+$ , the part 2. of associativity of twisted intertwining operators can be derived from the part 1.

We have finished the prove.  $\square$

The condition 8. in Theorem 11.1.1 is weaker than the following:

- Any twisted module generated by one element is an object of  $\mathcal{C}$ .

Therefore, we have

**Corollary 11.1.2.** *Suppose  $\mathcal{C}$  satisfies Condition 1-7 in Theorem 11.1.1. Furthermore, we assume that for any  $g \in \{h \in \text{Aut}(V) \mid \text{Obj}(\mathcal{C}_h) \neq \emptyset\}$ , any  $g$ -twisted module generated by one element is an object of  $\mathcal{C}$ . Then, the associativity of twisted intertwining operators for  $\mathcal{C}$  holds.*

The condition 8. in Theorem 11.1.1 is also weaker than the followings:

- Any grading-restricted twisted module generated by one element is an object of  $\mathcal{C}$ .
- There exists  $N \in \mathbb{N}$  such that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; \zeta_1, \zeta_2) = 0$ , for any  $n \in \mathbb{C}$  with  $\Re(n) < -N$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , and  $(\zeta_1, \zeta_2) \in F_2(\mathbb{C}^\times)$ .

Therefore, we have

**Corollary 11.1.3.** *Suppose  $\mathcal{C}$  satisfies Condition 1-7 in Theorem 11.1.1. Furthermore, we assume*

1. *For any  $g \in \{h \in \text{Aut}(V) \mid \text{Obj}(\mathcal{C}_h) \neq \emptyset\}$ , any grading-restricted  $g$ -twisted module generated by one element is an object of  $\mathcal{C}$ .*

2. Let  $W_1, W_2, W_3, M_1, \mathcal{Y}_1, \mathcal{Y}_2$  be as in condition 7. in Theorem 11.1.1. There exists  $N_{\mathcal{Y}_1, \mathcal{Y}_2} \in \mathbb{N}$  such that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) = 0$ , for any  $n \in \mathbb{C}$  with  $\Re(n) < -N_{\mathcal{Y}_1, \mathcal{Y}_2}$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , and  $(z_1, z_2)$  satisfying (11.1).

(See (9.4) for the definition of  $\tilde{\lambda}_n^{(2)}$ .)

Then, the associativity of twisted intertwining operators for  $\mathcal{C}$  holds.

Using Theorems 11.1.1 and 9.0.4, Corollaries 6.0.12, 7.0.24 and 11.1.3, and Remarks 7.0.19 and 8.0.2, we derive the following Corollary:

**Corollary 11.1.4.** *Let  $G$  be a subgroup of  $\text{Aut}(V)$ . Assume that the following conditions are satisfied:*

1. *For  $g \in G$ , there are only finitely many irreducible grading-restricted  $g$ -twisted  $V$ -modules.*
2. *For  $g \in G$ , every grading-restricted  $g$ -twisted  $V$ -module is completely reducible.*
3. *For  $g_1, g_2 \in G$ , and grading-restricted  $g_1^-$ ,  $g_2^-$ ,  $g_1 g_2$ -twisted  $V$ -modules  $W_1, W_2$ , and  $W_3$ , the fusion rule  $N_{W_1 W_2}^{W_3} := \dim \mathcal{V}_{W_1 W_2}^{W_3}$  is finite.*
4. *For any  $g \in G$ , and irreducible grading-restricted  $g$ -twisted  $V$ -module  $W$ ,  $L_W(0)_N$  is nilpotent.*
5. *Let  $W_1, W_2, W_3, M_1, M_2, W_4$  be any  $g_1^-$ ,  $g_2^-$ ,  $g_3^-$ ,  $g_2 g_3^-$ ,  $g_1 g_2^-$ ,  $g_1 g_2 g_3^-$ -twisted  $V$ -modules in  $\mathcal{C}$ , respectively. Let  $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$  be any twisted intertwining operators of types  $\begin{pmatrix} W_4 \\ W_1 & M_1 \end{pmatrix}$ ,  $\begin{pmatrix} M_1 \\ W_2 & W_3 \end{pmatrix}$ ,  $\begin{pmatrix} W_4 \\ M_2 & W_3 \end{pmatrix}$ ,  $\begin{pmatrix} M_2 \\ W_1 & W_2 \end{pmatrix}$ , respectively.*
  - (a)  $(\mathcal{Y}_1, \mathcal{Y}_2)$  *satisfies the convergence and extension property for products.*
  - (b)  $(\mathcal{Y}_3, \mathcal{Y}_4)$  *satisfies the convergence and extension property for iterates.*
  - (c) *There exists  $N_{\mathcal{Y}_1, \mathcal{Y}_2} \in \mathbb{N}$  such that  $\tilde{\lambda}_n^{(2)}(w_3, w'_4; z_1, z_2) = 0$ , for any  $n \in \mathbb{C}$  with  $\Re(n) < -N_{\mathcal{Y}_1, \mathcal{Y}_2}$ ,  $w_3 \in W_3$ ,  $w'_4 \in W'_4$ , and  $(z_1, z_2)$  satisfying (11.1).*  
(See (9.4) for the definition of  $\tilde{\lambda}_n^{(2)}$ .)

Then, the associativity of twisted intertwining operators for  $\mathcal{GM}_{gr}(G)$  holds.

## 11.2 Associativity isomorphisms of the $P(z)$ -tensor products

Using our results on the associativity of twisted intertwining operators, one can always get the corresponding result on associativity of  $P(z)$ -tensor products.

**Lemma 11.2.1.** *Suppose  $\mathcal{C}$  is a category of twisted  $V$ -modules such that for any  $z \in \mathbb{C}^\times$ , and any  $W_1$  and  $W_2$  in  $\mathcal{C}$ , the  $P(z)$ -tensor product  $W_1 \boxtimes_{P(z)} W_2$  of  $W_1$  and  $W_2$  in  $\mathcal{C}$  exists. The followings are equivalent:*

1. *The category  $\mathcal{C}$  satisfies the associativity of twisted intertwining operators.*
2. *For any  $z_1, z_2 \in \mathbb{C}$  satisfying (11.1), there exists a unique natural isomorphism*

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : - \boxtimes_{P(z_1)} (- \boxtimes_{P(z_2)} -) \Rightarrow (- \boxtimes_{P(z_1-z_2)} -) \boxtimes_{P(z_2)} -$$

such that for any twisted modules  $W_1, W_2$ , and  $W_3$  in  $\mathcal{C}$ , the map

$$\begin{aligned} \overline{\left( \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} \right)_{W_1, W_2, W_3}} &: \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \\ &\rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3} \end{aligned}$$

satisfies that for any  $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$ ,

$$\begin{aligned} \overline{\left( \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} \right)_{W_1, W_2, W_3}} (w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) \\ = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3. \end{aligned} \quad (11.6)$$

*Proof.* We only prove that 1. implies 2. because the other direction can be proved

by inverting the whole argument. Suppose  $\mathcal{C}$  satisfies the associativity of twisted intertwining operators.

Let  $\mathcal{Y}_{\boxtimes_{P(z_1)}}$  and  $\mathcal{Y}_{\boxtimes_{P(z_2)}}$  be twisted intertwining operators which induces the  $P(z_1)$ -, and  $P(z_2)$ -tensor product, with types  $\left(\begin{smallmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \quad W_2 \boxtimes_{P(z_2)} W_3 \end{smallmatrix}\right)$ ,  $\left(\begin{smallmatrix} W_2 \boxtimes_{P(z_2)} W_3 \\ W_2 \quad W_3 \end{smallmatrix}\right)$ , respectively. There exist a twisted module  $M_2$  in  $\mathcal{C}$ , and twisted intertwining operators  $\mathcal{Y}_3$  and  $\mathcal{Y}_4$  of types  $\left(\begin{smallmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ M_2 \quad W_3 \end{smallmatrix}\right)$ ,  $\left(\begin{smallmatrix} M_2 \\ W_1 W_2 \end{smallmatrix}\right)$  such that for any  $w_i \in W_i$ ,  $i = 1, 2, 3$ , and  $w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$ , we have

$$\langle w', \mathcal{Y}_{\boxtimes_{P(z_1)}}(w_1, z_1) \mathcal{Y}_{\boxtimes_{P(z_2)}}(w_2, z_2) w_3 \rangle = \langle w', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2) w_2, z_2) w_3 \rangle. \quad (11.7)$$

Notice that since  $\mathcal{Y}_4$  is a twisted intertwining operator,  $w_1 \otimes w_2 \mapsto \mathcal{Y}_4(w_1, z_1 - z_2) w_2$  is a  $P(z_1 - z_2)$ -product of  $W_1$  and  $W_2$ . By the universal property of  $P(z_1 - z_2)$ -tensor product, there uniquely exists a module map  $f : W_1 \boxtimes_{P(z_1 - z_2)} W_2 \rightarrow M$  such that  $\mathcal{Y}_4(w_1, z_1 - z_2) w_2 = \bar{f}(w_1 \boxtimes_{P(z_1 - z_2)} w_2)$  for any  $w_1 \in W_1$ ,  $w_2 \in W_2$ .

Let  $\mathcal{Y}_5(w, x) w_3 = \mathcal{Y}_3(f(w), x) w_3$  for any  $w \in W_1 \boxtimes_{P(z_1 - z_2)} W_2$  and  $w_3 \in W_3$ . Since  $f$  is a module map, we know that  $\mathcal{Y}_5$  is a twisted intertwining operator of type  $\left(\begin{smallmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \boxtimes_{P(z_1 - z_2)} W_2 \quad W_3 \end{smallmatrix}\right)$ . Since  $w \otimes w_3 \mapsto \mathcal{Y}_5(w, z_2) w_3$ ,  $w \in W_1 \boxtimes_{P(z_1 - z_2)} W_2$ ,  $w_3 \in W_3$  is a  $P(z_2)$ -product of  $W_1 \boxtimes_{P(z_1 - z_2)} W_2$  and  $W_3$ . By the universal property of  $P(z_2)$ -tensor product, there uniquely exists a module map

$$B : (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3 \rightarrow W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

such that  $\mathcal{Y}_5(w, z_2) w_3 = \bar{B}(w \boxtimes_{P(z_2)} w_3)$  for any  $w \in W_1 \boxtimes_{P(z_1 - z_2)} W_2$ ,  $w_3 \in W_3$ .

Therefore, for any  $w_i \in W_i$ ,  $i = 1, 2, 3$ , and  $w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$ , we have

$$\begin{aligned} & \langle w', (w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) \rangle \\ &= \langle w', \mathcal{Y}_{\boxtimes_{P(z_1)}}(w_1, z_1) \mathcal{Y}_{\boxtimes_{P(z_2)}}(w_2, z_2) w_3 \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle w', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle \\
&= \langle w', \mathcal{Y}_3(\bar{f}(w_1 \boxtimes_{P(z_1-z_2)} w_2), z_2)w_3 \rangle \\
&= \langle w', \mathcal{Y}_5(w_1 \boxtimes_{P(z_1-z_2)} w_2, z_2)w_3 \rangle \\
&= \langle w', \bar{B}((w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3) \rangle.
\end{aligned}$$

Therefore, we know that

$$\bar{B}((w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3) = w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3), \text{ for any } w_i \in W_i, i = 1, 2, 3. \quad (11.8)$$

Moreover, we know that (11.8) uniquely determines  $B$ .

Applying the second part of the associativity of twisted intertwining operators to the twisted intertwining operator  $\mathcal{Y}_{\boxtimes_{P(z_2)}}, \mathcal{Y}_{\boxtimes_{P(z_1-z_2)}}$  of types  $\left( \begin{smallmatrix} (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\ W_1 \boxtimes_{P(z_1-z_2)} W_2 \quad W_3 \end{smallmatrix} \right)$ ,  $\left( \begin{smallmatrix} W_1 \boxtimes_{P(z_1-z_2)} W_2 \\ W_1 \boxtimes_{P(z_1-z_2)} W_1 \quad W_2 \end{smallmatrix} \right)$ , we know there exist a twisted module  $M_1$  in  $\mathcal{C}$ , and twisted intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  of types  $\left( \begin{smallmatrix} (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\ W_1 M_1 \end{smallmatrix} \right)$ ,  $\left( \begin{smallmatrix} M_1 \\ W_2 W_3 \end{smallmatrix} \right)$ , such that

$$\langle w', \mathcal{Y}_{\boxtimes_{P(z_2)}}(\mathcal{Y}_{\boxtimes_{P(z_1-z_2)}}(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle = \langle w', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3 \rangle.$$

Using the universality property of  $P(z_2)$ -tensor product, there uniquely exists a module map  $g : W_2 \boxtimes_{P(z_2)} W_3 \rightarrow M_1$ , such that  $\bar{g}(w_2 \boxtimes_{P(z_2)} w_3) = \mathcal{Y}_2(w_2, z_2)w_3$  for any  $w_2 \in W_2, w_3 \in W_3$ . Let  $\mathcal{Y}_6(w_1, x)w = \mathcal{Y}_1(w_1, x)g(w)$  for any  $w_1 \in W_1, w \in W_2 \boxtimes_{P(z_2)} W_3$ . Since  $g$  is a module map, we know that  $\mathcal{Y}_6$  is a twisted intertwining map of type  $\left( \begin{smallmatrix} (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\ W_1 \quad W_2 \boxtimes_{P(z_2)} W_3 \end{smallmatrix} \right)$ . Since  $w_1 \otimes w \mapsto \mathcal{Y}_6(w_1, z_1)w$  is a  $P(z_1)$ -product of  $W_1$  and  $W_2 \boxtimes_{P(z_2)} W_3$ , we know that there uniquely exists a module map

$$A : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

such that  $\bar{A}(w_1 \boxtimes_{P(z_1)} w) = \mathcal{Y}_6(w_1, z_1)w$ , for any  $w_1 \in W_1, w \in W_2 \boxtimes_{P(z_2)} W_3$ .

Therefore, for any  $w_i \in W_i$ ,  $i = 1, 2, 3$ , and  $w' \in ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)'$ , we have

$$\begin{aligned}
& \langle w', (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \rangle \\
&= \langle w', \mathcal{Y}_{\boxtimes_{P(z_2)}}(\mathcal{Y}_{\boxtimes_{P(z_1-z_2)}}(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle \\
&= \langle w', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3 \rangle \\
&= \langle w', \mathcal{Y}_1(w_1, z_1)\bar{g}(w_2 \boxtimes_{P(z_2)} w_3) \rangle \\
&= \langle w', \mathcal{Y}_6(w_1, z_1)(w_2 \boxtimes_{P(z_2)} w_3) \rangle \\
&= \langle w', \bar{A}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) \rangle
\end{aligned}$$

Therefore, we know that

$$\bar{A}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3, \text{ for any } w_i \in W_i, i = 1, 2, 3. \quad (11.9)$$

Moreover, we know that (11.9) uniquely determines  $A$ .

By (11.8) and (11.9), it is clear that  $AB = \text{id}$  and  $BA = \text{id}$ . Therefore,  $A$  and  $B$  are isomorphisms. Let

$$\left( \mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} \right)_{W_1, W_2, W_3}.$$

It is not hard to see that  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  is functorial in all three positions. In other words,  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  is a natural transformation from the functor

$$- \boxtimes_{P(z_1)} (- \boxtimes_{P(z_2)} -) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

to the functor

$$(- \boxtimes_{P(z_1-z_2)} -) \boxtimes_{P(z_2)} - : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}.$$

The uniqueness of  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$  is easy.

□

As easy results of Lemma 11.2.1, Theorem 11.1.1, and Corollary 11.1.4, we have the following Corollaries:

**Corollary 11.2.2.** *Assume  $\mathcal{C}$  is a category of twisted modules satisfying all the conditions in Theorem 11.1.1, or Corollary 11.1.2, or Corollary 11.1.3. Then, we know that  $-\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-)$  and  $(-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$  can be viewed as two functors from  $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$  to  $\mathcal{C}$*

*For any  $z_1, z_2 \in \mathbb{C}$  satisfying (11.1), there exists a unique natural isomorphism*

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : -\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-) \Rightarrow (-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$$

*satisfying (11.6).*

**Corollary 11.2.3.** *Let  $G$  be a subgroup of  $\text{Aut}(V)$ . Assume all conditions in Corollary 11.1.4 are satisfied. Then, we know that  $-\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-)$  and  $(-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$  can be viewed as two functors from  $\mathcal{GM}_{gr}(G) \times \mathcal{GM}_{gr}(G) \times \mathcal{GM}_{gr}(G)$  to  $\mathcal{GM}_{gr}(G)$*

*For any  $z_1, z_2 \in \mathbb{C}$  satisfying (11.1), there exists a unique natural isomorphism*

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : -\boxtimes_{P(z_1)}(-\boxtimes_{P(z_2)}-) \Rightarrow (-\boxtimes_{P(z_1-z_2)}-)\boxtimes_{P(z_2)}-$$

*satisfying (11.6).*

# Appendix A

## A lemma on convergence

### A.0.1 Settings and Notations

Let  $A = (a_{i,j}) \in \text{GL}_n(\mathbb{C})$ ,  $b = (b_1, \dots, b_n) \in \mathbb{C}^n$ ,  $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ , and  $\delta = (\delta_1, \dots, \delta_n) \in \{0, \infty\}^n$ . Write  $u = (u_1, \dots, u_n) = (z_1, \dots, z_n)A - (b_1, \dots, b_n)$ , where each

$$u_i = u_i(z_1, \dots, z_n) = a_{1,i}z_1 + \dots + a_{n,i}z_n - b_i$$

is a function of  $z_1, \dots, z_n$ . In other words,  $u : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is an invertible affine transformation dependent on  $A$  and  $b$ .

Define the region

$$\Delta_{A,b,\delta}^\times(r) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |u_i| > r_i \text{ if } \delta_i = \infty, 0 < |u_i| < r_i \text{ if } \delta_i = 0\}. \quad (\text{A.1})$$

Clearly,  $\Delta_{A,b,\delta}^\times(r) = u^{-1}(\Delta_{I_n,0,\delta}^\times(r))$ . For the  $A, b, \delta$ , and  $r$  we fixed, we write  $M = \Delta_{A,b,\delta}^\times(r)$ . Also, define  $M_0$  to be the space such that  $M_0$  is set-theoretically equal to  $M$ , but with a different topology by adding cuttings along  $u_i \in \mathbb{R}_+$ ,  $i = 1, \dots, n$ . The positive real line in the  $u_i$ -plane, namely  $u_i \in \mathbb{R}_+$ , is attached to the upper half  $u_i$ -plane. In this way,  $M_0$  become simply-connected.

We define  $\Delta_{A,b,\delta}(r)$  in the following way. First, define

$$\Delta_{I_n,0,\delta}(r) = \left\{ (z_1, \dots, z_n) \in (\hat{\mathbb{C}})^n \mid |z_i| > r_i \text{ if } \delta_i = \infty, |z_i| < r_i \text{ if } \delta_i = 0 \right\},$$

where  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is the Riemann sphere and  $|\infty| = +\infty$ . (Note that  $\delta \in \Delta_{I_n,0,\delta}(r)$ .) The space  $\Delta_{I_n,0,\delta}(r)$  has a natural complex manifold structure as it is an open subset of  $(\hat{\mathbb{C}})^n$ . Consider the compactification

$$\hat{u} : \mathbb{C}^n \rightarrow \mathbb{C}^n \hookrightarrow (\hat{\mathbb{C}})^n : (z_1, \dots, z_n) \mapsto (u_1, \dots, u_n).$$

Let  $C(A, b)$  be a complex manifold and  $i : \mathbb{C}^n \hookrightarrow C(A, b)$  be a holomorphic embedding such that  $\hat{u}$  extends from  $\mathbb{C}^n \rightarrow (\hat{\mathbb{C}})^n$  to a biholomorphic map  $\hat{u} : C(A, b) \rightarrow (\hat{\mathbb{C}})^n$ . We know that such  $C(A, b)$  and  $i : \mathbb{C}^n \hookrightarrow C(A, b)$  are canonically unique. Define

$$\Delta_{A,b,\delta}(r) = \hat{u}^{-1}(\Delta_{I_n,0,\delta}(r)) \subset C(A, b),$$

which has a complex manifold structure given by  $\hat{u}$  and  $\Delta_{I_n,0,\delta}(r)$ . Note that  $\Delta_{A,b,\delta}^\times(r) \subset \Delta_{A,b,\delta}(r)$  and  $\hat{u}^{-1}(\delta) \in \Delta_{A,b,\delta}(r)$ . We write  $\hat{M} = \Delta_{A,b,\delta}(r)$ .

Suppose  $f(z_1, \dots, z_n)$  is a single valued analytic function defined on some region  $\Omega \subset \mathbb{C}^n$  containing  $M$  such that  $(z_1, \dots, z_n)A - b = \delta$  is a pole of  $f(z_1, \dots, z_n)$ . By ‘‘pole’’, what we mean is that there exists such a  $M$  given above,  $L \in -\mathbb{N}$ , and  $a_{k_1, \dots, k_n} \in \mathbb{C}$  for  $k_j \in \mathbb{Z}$ ,  $j = 1, \dots, n$ , such that

$$f(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n \in \mathbb{Z}} a_{k_1, \dots, k_n} u_1^{k_1} \cdots u_n^{k_n}, \quad (\text{A.2})$$

where the right-hand side is absolutely convergent on the region  $(z_1, \dots, z_n) \in M$ . Moreover,  $a_{k_1, \dots, k_n} \neq 0$  only if  $k_j \in L + \mathbb{N}$  for any  $j$  with  $\delta_j = 0$ , and  $k_{j'} \in -L - \mathbb{N}$  for any  $j'$  with  $\delta_{j'} = \infty$ . Clearly,  $f(z_1, \dots, z_n)$  is analytic at  $(z_1, \dots, z_n)A = \delta$  if and

only if  $L$  can be taken as 0.

Let  $\Omega \subset \mathbb{C}^n$  be a region on  $\mathbb{C}^n$ . We define the *maximal Reinhardt sub-region* of  $\Omega$  to be

$$R(\Omega) = \{(z_1, \dots, z_n) \in \Omega \mid (e^{\theta_1 i} z_1, \dots, e^{\theta_n i} z_n) \in \Omega \text{ for any } \theta_i \in \mathbb{R}, i = 1, \dots, n\}.$$

We say that  $\Omega$  is a *Reinhardt region* or *Reinhardt domain* if  $\Omega = R(\Omega)$ .

For holomorphic function with several complex variables, we have the following theorem:

**Lemma A.0.1.** *Let  $\Omega \subset \mathbb{C}^n$  be a connected Reinhardt domain containing the origin, and  $f \in \text{Hol}(\Omega)$ . Then there exists a unique power series such that  $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$  with normal convergence in  $\Omega$ . Moreover, we have  $a_\alpha = \frac{(\partial^\alpha f)(0)}{\alpha!}$ .*

We can generalize the condition of containing the origin to containing  $\delta$ . By the above lemma, we can easily prove the following.

**Lemma A.0.2.** *Let  $\Omega \subset \mathbb{C}^n$  be a region containing such a  $M$  given above, and  $f \in \text{Hol}(\Omega)$ . Then,  $(z_1, \dots, z_n)A - b = \delta$  is a pole of  $f$  if and only if for any  $K \gg 0$ ,*

$$\left( \prod_{i=1}^n u_i^{\epsilon_i} \right)^K \cdot f(z_1, \dots, z_n)$$

*can be analytically extended from  $M$  to  $\hat{M}$ , where  $\epsilon_i = 1, -1$  if  $\delta_i = 0, \infty$ , respectively.*

For  $A \in \text{GL}_n(\mathbb{C})$  and  $b \in \mathbb{C}^n$ , we also define

$$R(\Omega; A, b) = \{(z_1, \dots, z_n) \in \Omega \mid (e^{\theta_1 i} u_1, \dots, e^{\theta_n i} u_n) \in u(\Omega) \text{ for any } \theta_i \in \mathbb{R}, i = 1, \dots, n\},$$

where  $u = (u_1, \dots, u_n) : (z_1, \dots, z_n) \mapsto (z_1, \dots, z_n)A - b$  are the same as above. Using lemma 1.1, we can easily prove the following.

**Lemma A.0.3.** *Let  $\Omega \subset \mathbb{C}^n$  be a connected region containing such a  $M$  given above. For any  $f \in \text{Hol}(\Omega)$ , if (A.2) holds on  $(z_1, \dots, z_n) \in M$ , then it holds on the connected component of  $R(\Omega; A, b)$  containing  $M$ .*

Let  $\Omega \subset \mathbb{C}^n$  be a region on  $\mathbb{C}^n$ . We define

$$\begin{aligned} \bar{R}(\Omega; \delta) &= \left\{ (z_1, \dots, z_n) \in (\hat{\mathbb{C}})^n \mid \text{There exist } \lambda_i \in \hat{\mathbb{C}}, |\lambda_i|^{\epsilon_i} \leq 1, i = 1, \dots, n, \right. \\ &\quad \left. \text{and } (w_1, \dots, w_n) \in \Omega, \text{ such that } (z_1, \dots, z_n) = (\lambda_1 w_1, \dots, \lambda_n w_n) \cdot \right\}, \\ \bar{R}(\Omega; A, b, \delta) &= \hat{u}^{-1}(\bar{R}(\Omega; I_n, 0, \delta)) \subset C(A, b). \end{aligned}$$

Using lemma 1.1, we can easily prove the following.

**Lemma A.0.4.** *Let  $\Omega \subset \mathbb{C}^n$  be a region containing such a  $M$  given above. For any  $f \in \text{Hol}(\Omega)$ , if (A.2) holds on  $(z_1, \dots, z_n) \in M$ , then it holds on the connected component  $K$  of  $R(\Omega; A, b)$  containing  $M$ .*

*Moreover, if  $f$  can be analytically extended from  $M$  to  $\hat{M}$ , then, the right-hand side of (A.2) is absolutely convergent on the region  $\bar{R}(K; A, b, \delta)$ .*

For the following two sections, let  $A = I_n$ , and  $b = (0, \dots, 0)$ , which means  $M = \Delta_{I_n, 0, \delta}^\times(r)$ .

### A.0.2 Pole case

With  $A = I_n$ , and  $b = (0, \dots, 0)$ , the function  $f$  in the first section has expansion on  $(z_1, \dots, z_n) \in M = \Delta_{I_n, 0, \delta}^\times(r)$ ,

$$f(z_1, \dots, z_n) = \sum_{\substack{l_j \in \epsilon_j \cdot (L + \mathbb{N}) \\ j=1, \dots, n}} a_{l_1, \dots, l_n} z_1^{l_1} \cdots z_n^{l_n}, \tag{A.3}$$

where  $L$  is some non-positive integer and  $\epsilon_j$  is defined in lemma 1.2. Let  $K$  be the connected component of  $R(\Omega)$  containing  $\Delta_{I_n, 0, \delta}^\times(r)$ . By lemma A.0.4, (A.3) holds on

$K$ .

Now, let  $m \in \mathbb{Z}_+$ ,  $r' \in \mathbb{R}_+^m$ , and  $\delta^{(i)} \in \{0, \infty\}^m$ ,  $i = 1, \dots, n$ . Let  $\Omega_i \subset \mathbb{C}^m$ ,  $i = 1, \dots, n$  be connected regions containing  $\Delta_{I_m, 0, \delta^{(i)}}^\times(r')$ . Suppose for any  $i = 1, \dots, n$ , we have analytic functions

$$g_i(w_{(i),1}, \dots, w_{(i),m}) : \Omega_i \rightarrow \mathbb{C},$$

which has a pole at  $(w_{(i),1}, \dots, w_{(i),m}) = \delta^{(i)}$ . Therefore, for any  $i = 1, \dots, n$ , on the region  $(w_{(i),1}, \dots, w_{(i),m}) \in \Delta_{I_m, 0, \delta^{(i)}}^\times(r')$ , and therefore on  $K_i$ , the connected component of  $R(\Omega_i)$  containing  $\Delta_{I_m, 0, \delta^{(i)}}^\times(r')$ , we have the series expansions similar as (A.2),

$$g_i(w_{(i),1}, \dots, w_{(i),m}) = \sum_{\substack{k_j \in \epsilon_{(i),j}(L_i + \mathbb{N}) \\ j=1, \dots, n}} a_{k_1, \dots, k_n}^{(i)} w_{(i),1}^{k_1} \cdots w_{(i),n}^{k_n}, \quad (\text{A.4})$$

where  $L_i \in \mathbb{Z}$ , and  $\epsilon_{(i),j} = 1$  or  $-1$  if  $\delta_j^{(i)} = 0$  or  $\infty$ , respectively. We assume  $g_i$  satisfies the following condition:

$$\lim_{(w_{(i),1}, \dots, w_{(i),m}) \rightarrow \delta^{(i)}} g_i(w_{(i),1}, \dots, w_{(i),m}) = \delta_i, \text{ for any } i = 1, \dots, n. \quad (\text{A.5})$$

For any  $l \in \mathbb{Z}_+$ , on the region  $(w_{(i),1}, \dots, w_{(i),m}) \in K_i$ , there exists  $b_{k_1, \dots, k_m}^{(i)} \in \mathbb{C}$  such that

$$g_i(w_{(i),1}, \dots, w_{(i),m})^l = \sum_{\substack{k_j \in \epsilon_{(i),j}(L_i + \mathbb{N}) \\ j=1, \dots, m}} b_{k_1, \dots, k_m}^{(i)} w_{(i),1}^{k_1} \cdots w_{(i),m}^{k_m}, \quad (\text{A.6})$$

$$b_{k_1, \dots, k_m}^{(i)} = \sum_{j=1}^m \sum_{\substack{q_{1,j}, \dots, q_{l,j} \in \epsilon_{(i),j}(L_i + \mathbb{N}) \\ q_{1,j} + \dots + q_{l,j} = k_j}} \left( a_{q_{1,1}, \dots, q_{1,m}}^{(i)} \cdots a_{q_{l,1}, \dots, q_{l,m}}^{(i)} \right),$$

where the right-hand side of (A.6) is absolutely convergent on the region  $K_i$ .

However, to get an expansion like (A.6) for the  $l \in \mathbb{C}$  case, we need some conditions

for  $g_i$  to be satisfied. Define

$$k_j^{(i)} = \begin{cases} \min\{k_j \in \mathbb{Z} \mid a_{k_1, \dots, k_m}^{(i)} \neq 0 \text{ for some } k_l, l = 1, \dots, m, l \neq j\}, & \text{if } \delta_j^{(i)} = 0, \\ \max\{k_j \in \mathbb{Z} \mid a_{k_1, \dots, k_m}^{(i)} \neq 0 \text{ for some } k_l, l = 1, \dots, m, l \neq j\}, & \text{if } \delta_j^{(i)} = \infty. \end{cases}$$

We assume that  $a_{k_1^{(i)}, \dots, k_m^{(i)}}^{(i)} \neq 0$  for  $i = 1, \dots, n$ . WLOG, we assume  $a_{k_1^{(i)}, \dots, k_m^{(i)}}^{(i)} = 1$ . In this case, on the region  $(w_{(i),1}, \dots, w_{(i),m}) \in K_i$ , we have

$$g_i(w_{(i),1}, \dots, w_{(i),m}) = \left( \prod_{j=1}^m w_{(i),j}^{k_j^{(i)}} \right) (1 + P_i), \quad (\text{A.7})$$

$$\text{where } P_i = P_i(w_{(i),1}, \dots, w_{(i),m}) \in \mathbb{C}[[w_{(i),1}^{\epsilon_{(i),1}}, \dots, w_{(i),m}^{\epsilon_{(i),m}}]]. \quad (\text{A.8})$$

We know that the constant term  $P_i$  is zero. Also,  $P_i$  is absolutely convergent on the region given by  $(w_{(i),1}, \dots, w_{(i),m}) \in K_i$ . Notice that the condition (A.5) is equivalent to either of the followings.

- $g_i$  can be analytically extended to a holomorphic map  $(\hat{\mathbb{C}})^m \supset \Delta_{I_m, 0, \delta^{(i)}}(r') \rightarrow \hat{\mathbb{C}}$ , and  $g_i(\delta^{(i)}) = \delta_i$ . (A.9)

- For any  $j \in \{1, \dots, m\}$ , we have  $\epsilon_i \cdot \epsilon_{(i),j} \cdot k_j^{(i)} \geq 0$ , and  $\sum_{j=1}^m \epsilon_{(i),j} \cdot k_j^{(i)} \neq 0$ . (A.10)

Now, regarding  $P_i$  as formal series, for any  $\alpha \in \mathbb{C}$ , there exist  $b_{k_1, \dots, k_m; \alpha}^{(i)}, c_{k_1, \dots, k_m}^{(i)} \in \mathbb{C}$  such that, as formal series, we have

$$\begin{aligned} \log(1 + P_i) &= \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l+1}}{l} (P_i)^l = \sum_{\substack{k_j \in \epsilon_{(i),j} \mathbb{Z}_+ \\ j=1, \dots, m}} c_{k_1, \dots, k_m}^{(i)} w_{(i),1}^{k_1} \dots w_{(i),m}^{k_m} \quad (\text{A.11}) \\ \exp(\alpha \log(1 + P_i)) &= \sum_{k \in \mathbb{N}} \frac{\alpha^k}{k!} \left( \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l+1}}{l} (P_i)^l \right)^k \end{aligned}$$

$$= \sum_{\substack{k_j \in \epsilon^{(i),j} \mathbb{N} \\ j=1,\dots,m}} b_{k_1+\alpha k_1^{(i)},\dots,k_m+\alpha k_m^{(i)};\alpha}^{(i)} w_{(i),1}^{k_1} \cdots w_{(i),m}^{k_m}. \quad (\text{A.12})$$

We define the formal series for  $\alpha \in \mathbb{C}$ ,

$$g_i(w_{(i),1}, \dots, w_{(i),m})^\alpha = \sum_{\substack{k_j \in \epsilon^{(i),j} \mathbb{N} \\ j=1,\dots,m}} b_{k_1+\alpha k_1^{(i)},\dots,k_m+\alpha k_m^{(i)};\alpha}^{(i)} w_{(i),1}^{k_1+\alpha k_1^{(i)}} \cdots w_{(i),m}^{k_m+\alpha k_m^{(i)}}.$$

For any  $(j_1, \dots, j_m) \in \mathbb{N}^m$ , let  $\partial_{(j_1, \dots, j_m)} = \frac{\partial^{j_1+\dots+j_m}}{(\partial x_1)^{j_1} \cdots (\partial x_m)^{j_m}}$  be a formal differential operator. Write

$$\partial_{(j_1, \dots, j_m)} (\exp(P(x_1, \dots, x_m))) = \exp\left(P(x_1, \dots, x_m)\right) \cdot B_{(j_1, \dots, j_m)}(\partial_\beta(P), \beta \in \mathbb{N}^m), \quad (\text{A.13})$$

where  $P(x_1, \dots, x_m) \in \mathbb{C}[[x_1, \dots, x_m]]$  is any formal power series with constant term zero, and  $B_{(j_1, \dots, j_m)}(\partial_\beta P, \beta \in \mathbb{N}^m) \in \mathbb{C}[\partial_\beta P, \beta \in \mathbb{N}^m]$  is a polynomial on the derivatives of  $P$ . Let  $P = \alpha \log(1 + P_i)$ ,  $x_j = w_{(i),j}^{\epsilon^{(i),j}}$ . Using these notations, we can write down the following relation

$$\begin{aligned} & b_{k_1+\alpha k_1^{(i)},\dots,k_m+\alpha k_m^{(i)};\alpha}^{(i)} \\ &= (|k_1|! \cdots |k_m|!) \partial_{(|k_1|, \dots, |k_m|)} \left( \exp\left(\alpha \log(1 + P_i(w_{(i),1}, \dots, w_{(i),m}))\right) \right) (\delta^{(i)}) \\ &= (|k_1|! \cdots |k_m|!) \cdot \exp\left(\alpha \log(1 + P_i)(\delta^{(i)})\right) \cdot B_{(|k_1|, \dots, |k_m|)}((\partial_\beta P)(\delta^{(i)}), \beta \in \mathbb{N}^m) \\ &= (|k_1|! \cdots |k_m|!) \cdot 1 \cdot B_{(|k_1|, \dots, |k_m|)}\left(\alpha \cdot \beta! \cdot c_{\epsilon^{(i),j} \cdot \beta_1, \dots, \epsilon^{(i),j} \cdot \beta_m}, \beta \in \mathbb{N}^m\right) \end{aligned} \quad (\text{A.14})$$

Let  $\tilde{l}_0(z)$ ,  $z \in \mathbb{C}^\times$  be the single valued branch of  $\text{Log}(z)$  such that  $\Im(\tilde{l}_0(z)) \in (-\pi, \pi]$  for any  $z \in \mathbb{C}^\times$ . We know that for any  $z \in \mathbb{C}$ ,  $|z| < 1$ ,

$$\tilde{l}_0(1+z) = \sum_{l \in \mathbb{Z}_+} \frac{(-1)^{l+1}}{l} z^l, \quad (\text{A.15})$$

where the right-hand side is absolutely convergent.

By lemma A.0.4, we know that the series  $P_i$  is absolutely convergent on  $\bar{R}(K_i; \delta^{(i)})$ . We define

$$U'_i = \left\{ (w_{(i),1}, \dots, w_{(i),m}) \in \bar{R}(K_i; \delta^{(i)}) \mid |P_i(w_{(i),1}, \dots, w_{(i),m})| < 1 \right\}.$$

Since  $P_i(\delta^{(i)}) = 0$  and  $\Delta_{I_m,0,\delta^{(i)}}^\times(r'') \subset K_i$ , for sufficiently small  $r''$ , we know that  $\Delta_{I_m,0,\delta^{(i)}}(r'') \subset U'_i$ . Let  $U_i$  be the connected component of  $U'_i$  containing  $\Delta_{I_m,0,\delta^{(i)}}(r'')$ .

**Lemma A.0.5.** *The right-hand side of (A.11) is absolutely convergent, on the region  $\bar{R}(R(U_i); \delta^{(i)})$ , to the analytic function  $\tilde{l}_0(1 + P_i)$ . The right-hand side of (A.12) is absolutely convergent, on the region  $\bar{R}(R(U_i); \delta^{(i)})$ , to the analytic function*

$$\exp\left(\alpha \tilde{l}_0(1 + P_i)\right).$$

*Proof.* We only prove the second half of this theorem, because the first half is similar and easier. To prove (A.12) is absolutely convergent on  $\bar{R}(R(U_i); \delta^{(i)})$ , by lemma A.0.4, we only need to prove that (A.12) is convergent on  $R(U_i)$ .

We know that  $\varphi_\alpha^{(i)}(w_{(i),1}, \dots, w_{(i),m}) = \exp\left(\alpha \tilde{l}_0(1 + P_i)\right)$  is a single valued analytic function on the region  $U'_i$  which contains some  $\Delta_{I_m,0,\delta^{(i)}}(r'')$ . Therefore, we know that there exist  $b_{k_1, \dots, k_m; \alpha}^{(i)} \in \mathbb{C}$  such that on the region  $R(U_i)$ ,

$$\varphi_\alpha^{(i)}(w_{(i),1}, \dots, w_{(i),m}) = \exp\left(\alpha \tilde{l}_0(1 + P_i)\right) = \sum_{\substack{k_j \in \epsilon_{(i),j} \mathbb{N} \\ j=1, \dots, m}} b_{k_1 + \alpha k_1^{(i)}, \dots, k_m + \alpha k_m^{(i)}; \alpha}^{(i)} w_{(i),1}^{k_1} \cdots w_{(i),m}^{k_m},$$

where the right-hand side is absolutely convergent on the region  $\bar{R}(R(U_i); \delta^{(i)})$ . Now, we need to prove that  $b_{k_1, \dots, k_m; \alpha}^{(i)} = b_{k_1 + \alpha k_1^{(i)}, \dots, k_m + \alpha k_m^{(i)}; \alpha}^{(i)}$ .

By lemma 1.1, we know that

$$b'_{k_1+\alpha k_1^{(i)}, \dots, k_m+\alpha k_m^{(i)}; \alpha}^{(i)} = \frac{1}{|k_1|! \cdots |k_m|!} \cdot (\partial^{(k_1, \dots, k_m)} \varphi_\alpha^{(i)}) (\delta^{(i)}),$$

$$\text{where } \partial^{(k_1, \dots, k_m)} = \frac{\partial^{|k_1|+\dots+|k_m|}}{\left(\partial w_{(i),1}^{\epsilon^{(i),1}}\right)^{|k_1|} \cdots \left(\partial w_{(i),m}^{\epsilon^{(i),m}}\right)^{|k_m|}}. \quad (\text{A.16})$$

Since  $\exp(z) = \sum_{k \geq 0} \frac{z^k}{k!}$  uniformly absolutely converges on any compact subset of  $\mathbb{C}$ , we know that  $\sum_{k \geq 0} \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!}$  uniformly absolutely converges on any compact subset of  $U_i$ . Also, for any  $j = 1, \dots, n$ , the series  $\sum_{k \geq 0} \frac{\partial}{\partial w_{(i),j}^{\epsilon^{(i),j}}} \left( \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!} \right) = \frac{\partial(\alpha \tilde{l}_0(1+P_i))}{\partial w_{(i),j}^{\epsilon^{(i),j}}}$ .  $\sum_{k \geq 0} \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!}$  is also uniformly absolutely convergent on any compact subset of  $U_i$ . Therefore, we know that  $\frac{\partial}{\partial w_{(i),j}^{\epsilon^{(i),j}}} \left( \sum_{k \geq 0} \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!} \right) = \sum_{k \geq 0} \frac{\partial}{\partial w_{(i),j}^{\epsilon^{(i),j}}} \left( \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!} \right)$ . Use induction, we know that  $\partial^{(k_1, \dots, k_m)} \left( \sum_{k \geq 0} \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!} \right) = \sum_{k \geq 0} \partial^{(k_1, \dots, k_m)} \left( \frac{(\alpha \tilde{l}_0(1+P_i))^k}{k!} \right)$ . Therefore, we know that

$$\begin{aligned} & \partial^{(k_1, \dots, k_m)} (\varphi_\alpha^{(i)}) \\ &= \exp\left(\alpha \tilde{l}_0(1+P_i)\right) \cdot B_{(|k_1|, \dots, |k_m|)}\left(\partial^{(\epsilon^{(i),1}\beta_1, \dots, \epsilon^{(i),m}\beta_m)}(\alpha \tilde{l}_0(1+P_i)), \beta \in \mathbb{N}^m\right) \end{aligned} \quad (\text{A.17})$$

Similarly, since  $\tilde{l}_0(1+z) = \sum_{k > 0} \frac{(-1)^{k+1}}{k} z^k$  uniformly absolutely converges on any compact subset of  $\{z \in \mathbb{C} \mid |z| < 1\}$ , we know that  $\sum_{k \geq 0} \frac{(-1)^{k+1}}{k} P_i^k$  uniformly absolutely converges on any compact subset of  $U_i$ . Also, for any  $j = 1, \dots, n$ , the series  $\sum_{k \geq 0} \frac{\partial}{\partial w_{(i),j}^{\epsilon^{(i),j}}} \left( \frac{(-1)^{k+1}}{k} P_i^k \right) = \frac{\partial P_i}{\partial w_{(i),j}^{\epsilon^{(i),j}}} \cdot \sum_{k \geq 0} (-1)^{k+1} P_i^k$  is also uniformly absolutely convergent on any compact subset of  $U_i$ . Use induction, we know that

$$\partial^{(\gamma_1, \dots, \gamma_m)} \left( \sum_{k \geq 0} \frac{(-1)^{k+1}}{k} P_i^k \right) = \sum_{k \geq 0} \partial^{(\gamma_1, \dots, \gamma_m)} \left( \frac{(-1)^{k+1}}{k} P_i^k \right),$$

for any  $\gamma_j \in \epsilon_{(i),j}\mathbb{N}$ . Therefore, by (A.15) we know that

$$\begin{aligned} \partial^{(\gamma_1, \dots, \gamma_m)} \left( \alpha \tilde{l}_0(1 + P_i) \right) (\delta^{(i)}) &= \alpha \cdot \frac{\partial^{|\gamma_1| + \dots + |\gamma_m|} \left( \sum_{k \geq 0} \frac{(-1)^{k+1}}{k} P_i^k \right)}{\left( \partial w_{(i),1}^{\epsilon_{(i),1}} \right)^{|\gamma_1|} \cdots \left( \partial w_{(i),m}^{\epsilon_{(i),m}} \right)^{|\gamma_m|}} (\delta^{(i)}) \\ &= \alpha \cdot (|\gamma_1|! \cdots |\gamma_m|!) \cdot c_{\gamma_1 + k_1^{(i)}, \dots, \gamma_m + k_m^{(i)}}^{(i)}. \end{aligned} \quad (\text{A.18})$$

Therefore, by (A.14), (A.16), (A.17), and (A.18), we know that

$$\begin{aligned} b_{k_1 + \alpha k_1^{(i)}, \dots, k_m + \alpha k_m^{(i)}; \alpha}^{(i)} &= \frac{1}{|k_1|! \cdots |k_m|!} \cdot (\partial^{(k_1, \dots, k_m)} \varphi_\alpha^{(i)}) (\delta^{(i)}), \\ &= \frac{1}{|k_1|! \cdots |k_m|!} \cdot \exp \left( \alpha \tilde{l}_0(1 + P_i) \right) (\delta^{(i)}) \cdot \\ &\quad B_{(|k_1|, \dots, |k_m|)} \left( \partial^{(\epsilon_{(i),1}\beta_1, \dots, \epsilon_{(i),m}\beta_m)} (\alpha \tilde{l}_0(1 + P_i)) (\delta^{(i)}), \beta \in \mathbb{N}^m \right) \\ &= \frac{1}{|k_1|! \cdots |k_m|!} \cdot 1 \cdot B_{(|k_1|, \dots, |k_m|)} \left( \alpha \cdot \beta! \cdot c_{\epsilon_{(i),j} \cdot \beta_j + k_1^{(i)}, \dots, \epsilon_{(i),j} \cdot \beta_j + k_m^{(i)}}^{(i)}, \beta \in \mathbb{N}^m \right) \\ &= b_{k_1 + \alpha k_1^{(i)}, \dots, k_m + \alpha k_m^{(i)}; \alpha}^{(i)}. \end{aligned} \quad (\text{A.19})$$

□

Since  $l_0(z)$  exists on  $\mathbb{C}^\times = \hat{\mathbb{C}} - \{0, \infty\}$ , we have the following lemma.

**Corollary A.0.6.** *Let  $p_i = p_i(w_{(i),1}, \dots, w_{(i),m}) \in \mathbb{Z}$  such that*

$$l_{p_i} \left( g_i(w_{(i),1}, \dots, w_{(i),m}) \right) = \tilde{l}_0(1 + P_i) + \sum_{j=1}^m k_j^{(i)} l_0(w_{(i),j}).$$

Then, for any  $\alpha \in \mathbb{C}$ ,

$$\sum_{j=1}^m k_j^{(i)} l_0(w_{(i),j}) + \sum_{\substack{k_j \in \epsilon_{(i),j}\mathbb{Z}_+ \\ j=1, \dots, m}} c_{k_1, \dots, k_m}^{(i)} w_{(i),1}^{k_1} \cdots w_{(i),m}^{k_m}, \quad (\text{A.20})$$

and

$$\begin{aligned}
& \left( \prod_{j=1}^m e^{\alpha k_j^{(i)} l_0(w_{(i),j})} \right) \cdot \sum_{\substack{k_j \in \epsilon_{(i),j} \mathbb{N} \\ j=1, \dots, n}} b_{k_1 + \alpha k_1^{(i)}, \dots, k_m + \alpha k_m^{(i)}; \alpha}^{(i)} w_{(i),1}^{k_1} \cdots w_{(i),m}^{k_m} \\
&= \sum_{\substack{k_j \in \epsilon_{(i),j} \mathbb{N} \\ j=1, \dots, m}} b_{k_1 + \alpha k_1^{(i)}, \dots, k_m + \alpha k_m^{(i)}; \alpha}^{(i)} e^{(k_1 + \alpha k_1^{(i)}) l_0(w_{(i),1})} \cdots e^{(k_m + \alpha k_m^{(i)}) l_0(w_{(i),m})}. \tag{A.21}
\end{aligned}$$

are absolutely convergent, on the region  $\bar{R}(R(U_i); \delta^{(i)}) \cap (\mathbb{C}^\times)^m$ , to

$$l_{p_i}(g_i(w_{(i),1}, \dots, w_{(i),m})), \text{ and } e^{\alpha l_{p_i}(g_i(w_{(i),1}, \dots, w_{(i),m}))}, \text{ respectively.}$$

Now, we consider the composition of  $f(z_1, \dots, z_n)$  and  $z_i = g_i(w_{(i),1}, \dots, w_{(i),m})$ ,  $i = 1, \dots, n$ . First, we do the following formal calculus without taking any convergence issues into consideration.

$$\begin{aligned}
f(g_1, \dots, g_n) &= \sum_{\substack{\ell_c \in \epsilon_c \cdot (L + \mathbb{N}) \\ c=1, \dots, n}} a_{\ell_1, \dots, \ell_n} g_1^{\ell_1} \cdots g_n^{\ell_n} \\
&= \sum_{\substack{\ell_c \in \epsilon_c \cdot (L + \mathbb{N}) \\ c=1, \dots, n}} a_{\ell_1, \dots, \ell_n} \left( \prod_{i=1}^n \sum_{\substack{k_j \in \epsilon_{(i),j} \mathbb{N} \\ j=1, \dots, m}} b_{k_1 + \ell_i k_1^{(i)}, \dots, k_m + \ell_i k_m^{(i)}; \ell_i}^{(i)} w_{(i),1}^{k_1 + \ell_i k_1^{(i)}} \cdots w_{(i),m}^{k_m + \ell_i k_m^{(i)}} \right) \\
&= \sum_{\substack{\ell_c \in \epsilon_c \cdot (L + \mathbb{N}) \\ c=1, \dots, n}} a_{\ell_1, \dots, \ell_n} \left( \sum_{\substack{k_{j,i} \in \epsilon_{(i),j} \mathbb{N} + \ell_i k_j^{(i)} \\ i=1, \dots, n, j=1, \dots, m}} \prod_{i'=1}^n b_{k_1, i', \dots, k_m, i'; \ell_{i'}}^{(i')} w_{(i'),1}^{k_1, i'} \cdots w_{(i'),m}^{k_m, i'} \right) \tag{A.22}
\end{aligned}$$

Notice that by (A.10), for any  $i = 1, \dots, n$ , there exists  $j_i \in \{1, \dots, m\}$  such that

$$\epsilon_i \cdot \epsilon_{(i),j_i} \cdot k_{j_i}^{(i)} \geq 1.$$

So, we know that for any constant  $k \in \mathbb{Z}$ , the set

$$\left\{ \ell_i \in \epsilon_i(L + \mathbb{N}) \mid k \in \epsilon_{(i),j} \mathbb{N} + \ell_i k_{j,i}^{(i)} \right\}$$

is a finite set. The finiteness of these sets implies that (A.22) is summable as a formal series on  $w_{(i),j}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . In other words, there exists  $b_k \in \mathbb{C}$  for any  $k = (k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})$ , such that the formal series (A.22) is equal to

$$\sum_{k=(k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})} \left( b_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right). \quad (\text{A.23})$$

By (A.10), we know that  $b_k = 0$  if  $k_{j,i} \ll 0$  for  $\epsilon_{(i),j} = -1$ , or  $k_{j,i} \gg 0$  for  $\epsilon_{(i),j} = 1$  for some  $i, j$ . In other words,  $(w_{(1),1}, \dots, w_{(1),m}, \dots, w_{(n),1}, \dots, w_{(n),m}) = (\delta^{(1)}, \dots, \delta^{(n)})$  is a pole of (A.23). For sufficiently large positive integer  $N$  (actually  $N > -L$  is enough), we do the same formal calculus for  $(\prod_{i=1}^n g_i^{\epsilon_i})^N f(g_1, \dots, g_n)$ , and have

$$\left( \prod_{i=1}^n g_i^{N\epsilon_i} \right) \cdot f(g_1, \dots, g_n) = \sum_{\substack{\ell_c \in \epsilon_c \cdot (L + \mathbb{N}) \\ c=1, \dots, n}} a_{\ell_1, \dots, \ell_n} g_1^{\ell_1 + N\epsilon_1} \dots g_n^{\ell_n + N\epsilon_n} \quad (\text{A.24})$$

$$\begin{aligned} &= \sum_{\substack{\ell_c \in \epsilon_c \cdot (L + \mathbb{N}) \\ c=1, \dots, n}} a_{\ell_1, \dots, \ell_n} \left( \sum_{\substack{k_{j,i} \in \epsilon_{(i),j} \mathbb{N} + (\ell_i + N\epsilon_i) k_j^{(i)} \\ i=1, \dots, n, j=1, \dots, m}} \prod_{i'=1}^n b_{k_{1,i'}, \dots, k_{m,i'}; \ell_{i'} + N\epsilon_{i'}} w_{(i'),1}^{k_{1,i'}} \dots w_{(i'),m}^{k_{m,i'}} \right) \\ &= \sum_{k=(k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})} \left( c_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right). \quad (\text{A.25}) \end{aligned}$$

For those  $i$  with  $\delta_i = 0$ , there are only non-negative powers of  $g_i$  in (A.24). For any  $l \in \mathbb{N}$ , the lowest (for  $\epsilon_{(i),j} = 1$ ), or highest (for  $\epsilon_{(i),j} = -1$ ) power of  $w_{(i),j}$  in the expansion of  $g_i^l$  is  $l \cdot k_j^{(i)}$ , which is zero or has the same sign as  $\epsilon_{(i),j}$ . For those  $i$  with  $\delta_i = \infty$ , and  $l \in -\mathbb{N}$ , we have similar result for  $g_i^l$ . Therefore, we know that  $c_k = 0$  unless  $k_{j,i} \in \epsilon_{(i),j} \mathbb{N}$  for all  $i = 1, \dots, n$ , and  $j = 1, \dots, m$ .

Also, we know that

$$c_k = \left( \prod_{i,j} |k_{j,i}|! \right)^{-1} \cdot \sum_{\substack{\ell_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1,\dots,n}} a_{\ell_1,\dots,\ell_n} \cdot \left( \sum_{\substack{k_{j,i} \in \epsilon_{(i),j} \mathbb{N} + (\ell_i + N\epsilon_i) k_j^{(i)} \\ i=1,\dots,n, j=1,\dots,m}} \prod_{i'=1}^n b_{k_{1,i'},\dots,k_{m,i'}; \ell_{i'} + N\epsilon_{i'}}^{(i')} w_{(i'),1}^{k_{1,i'}} \cdots w_{(i'),m}^{k_{m,i'}} \right) \quad (\text{A.26})$$

All the procedures to find  $b_k$  and  $c_k$  above are only formal calculus. Now we consider the analytic perspective and absolute convergence of (A.23).

**Theorem A.0.7.** *the series (A.23) is absolutely convergent to*

$$f(g_1(w_{(1),1}, \dots, w_{(1),m}), \dots, g_n(w_{(n),1}, \dots, w_{(n),m}))$$

on the connected component  $\tilde{K}$  of the region

$$\tilde{\Omega} = R \left( \left\{ (w_{(1),1}, \dots, w_{(1),m}, \dots, w_{(n),1}, \dots, w_{(n),m}) \in (\mathbb{C}^\times)^{mn} \mid \begin{aligned} & (w_{(i),1}, \dots, w_{(i),m}) \in \bar{R}(R(U_i); \delta^{(i)}) \cap (\mathbb{C}^\times)^m, \quad i = 1, \dots, n, \\ & (g_1(w_{(1),1}, \dots, w_{(1),m}), \dots, g_n(w_{(n),1}, \dots, w_{(n),m})) \in \bar{R}(K; \delta). \end{aligned} \right\} \right) \quad (\text{A.27})$$

containing  $\Delta_{I_m, 0, \delta^{(1)}}^\times(r'') \times \cdots \times \Delta_{I_m, 0, \delta^{(n)}}^\times(r'')$  for sufficiently small  $r''$ . (Notice that  $\tilde{K}$  exists and is non-empty.)

*Proof.* Since for sufficiently large positive integer  $N$ , the function

$$\left( \prod_{i=1}^n z_i^{\epsilon_i} \right)^N f(z_1, \dots, z_n)$$

is analytic on  $\bar{R}(K; \delta)$ , we know that

$$G(w_{(1),1}, \dots, w_{(1),m}, \dots, w_{(n),1}, \dots, w_{(n),m}) = \left( \prod_{i=1}^n g_i(w_{(i),1}, \dots, w_{(i),m})^{N\epsilon_i} \right) \cdot f(g_1(w_{(1),1}, \dots, w_{(1),m}), \dots, g_n(w_{(n),1}, \dots, w_{(n),m})) \quad (\text{A.28})$$

is analytic on the region given by  $(g_1(w_{(1),1}, \dots, w_{(1),m}), \dots, g_n(w_{(n),1}, \dots, w_{(n),m})) \in \bar{R}(K; \delta)$ . Since, by (A.9), we have  $(g_1(\delta^{(1)}), \dots, g_n(\delta^{(n)})) = \delta \in \Delta_{I_n, 0, \delta}(r) \subset \bar{R}(K; \delta)$ , we know that  $(\delta^{(1)}, \dots, \delta^{(n)})$  is contained in the above region, which means that (A.28) is analytic near  $(\delta^{(1)}, \dots, \delta^{(n)})$ . Therefore, we have the expansion of (A.28) at this point, i.e. on  $\tilde{K}$ , (A.28) is equal to an absolutely convergent series

$$\sum_{k=(k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})} \left( c'_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right), \text{ where } c'_k = 0 \text{ unless } k_{j,i} \in \epsilon_{(i),j} \mathbb{N} \text{ for all } i, j. \quad (\text{A.29})$$

First, we show that  $c'_k = c_k$ , for all  $k \in \text{Mat}_{m \times n}(\mathbb{Z})$ .

By lemma 1.1, we know that

$$c'_k = \left( \prod_{i,j} |k_{j,i}|! \right)^{-1} \cdot (\partial_k G)(\delta^{(1)}, \dots, \delta^{(n)}),$$

$$\text{where } \partial_k = \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} \frac{\partial^{|k_{j,i}|}}{\partial (w_{(i),j}^{\epsilon_{(i),j}})^{|k_{j,i}|}}. \quad (\text{A.30})$$

For any  $i = 1, \dots, n$ , and  $p \in \mathbb{N}$ , the series

$$\sum_{\substack{l_c \in \epsilon_j \cdot (L + \mathbb{N}) \\ c=1, \dots, n}} \frac{\partial^p}{\partial (z_i^{\epsilon_i})^p} (a_{l_1, \dots, l_n} z_1^{l_1 + N\epsilon_1} \dots z_n^{l_n + N\epsilon_n})$$

is uniformly absolutely convergent on any compact subset of  $\bar{R}(K; \delta)$ , which is because

the  $p = 0$  case holds. Therefore, the series

$$\begin{aligned} & \sum_{\substack{l_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1, \dots, n}} \frac{\partial}{\partial w_{(i),j}^{\epsilon_{(i),j}}} (a_{l_1, \dots, l_n} g_1^{l_1+N\epsilon_1} \dots g_n^{l_n+N\epsilon_n}) \\ &= \frac{\partial g_i^{\epsilon_i}}{\partial w_{(i),j}^{\epsilon_{(i),j}}} \cdot \sum_{\substack{l_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1, \dots, n}} \left[ \frac{\partial}{\partial z_i^{\epsilon_i}} (a_{l_1, \dots, l_n} z_1^{l_1+N\epsilon_1} \dots z_n^{l_n+N\epsilon_n}) \right] \Big|_{z_i=g_i} \end{aligned} \quad (\text{A.31})$$

is uniformly absolutely convergent on any compact subset of the region given by

$(g_1(w_{(1),1}, \dots, w_{(1),m}), \dots, g_n(w_{(n),1}, \dots, w_{(n),m})) \in \bar{R}(K; \delta)$ . Therefore, for any  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , we have

$$\frac{\partial G}{\partial w_{(i),j}^{\epsilon_{(i),j}}} = \sum_{\substack{l_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1, \dots, n}} \frac{\partial}{\partial w_{(i),j}^{\epsilon_{(i),j}}} (a_{l_1, \dots, l_n} g_1^{l_1+N\epsilon_1} \dots g_n^{l_n+N\epsilon_n}).$$

Similarly, for any  $k = (k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})$  with  $k_{j,i} \in \epsilon_{(i),j}\mathbb{N}$ , we have

$$\partial_k G = \sum_{\substack{l_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1, \dots, n}} \partial_k (a_{l_1, \dots, l_n} g_1^{l_1+N\epsilon_1} \dots g_n^{l_n+N\epsilon_n}).$$

Therefore, we know that

$$\begin{aligned} & \left( \prod_{i,j} |k_{j,i}|! \right)^{-1} (\partial_k G)(\delta^{(1)}, \dots, \delta^{(n)}) \\ &= \left( \prod_{i,j} |k_{j,i}|! \right)^{-1} \sum_{\substack{l_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1, \dots, n}} \partial_k (a_{l_1, \dots, l_n} g_1^{l_1+N\epsilon_1} \dots g_n^{l_n+N\epsilon_n}) (\delta^{(1)}, \dots, \delta^{(n)}) \\ &= \left( \prod_{i,j} |k_{j,i}|! \right)^{-1} \sum_{\substack{l_c \in \epsilon_c \cdot (L+\mathbb{N}) \\ c=1, \dots, n}} \partial_k \Big|_{\substack{w_{(i),j}^{\epsilon_{(i),j}}=0, \\ i=1, \dots, n, j=1, \dots, m.}} (a_{l_1, \dots, l_n} g_1^{l_1+N\epsilon_1} \dots g_n^{l_n+N\epsilon_n}) \end{aligned} \quad (\text{A.32})$$

Notice that on the non-empty region  $\tilde{K}$ ,  $g_i^\alpha$  ( $\alpha \in \mathbb{C}$ ) has an absolutely convergent series expansion stated in corollary A.0.6, which is used to define  $b_k$  and  $c_k$ . Therefore,

we can see that the right-hand side of (A.32) is equal to the right-hand side of (A.26).

By (A.30), we know that  $c'_k = c_k$  for any  $k \in \text{Mat}_{m \times n}(\mathbb{Z})$ .

By lemma A.0.5, we know that on  $\tilde{K}$ ,  $(\prod_{i=1}^n g_i^{-N\epsilon_i})$  can be written as an absolutely convergent series using corollary A.0.6. Therefore, on  $\tilde{K}$ , we have

$$\begin{aligned}
& \sum_{k=(k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})} \left( b_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right) \\
&= \left( \sum_{\substack{k_{j,i} \in \epsilon_{(i),j} \mathbb{N} - N\epsilon_i k_j^{(i)} \\ i=1, \dots, n, j=1, \dots, m}} \prod_{i'=1}^n b_{k_{1,i'}, \dots, k_{m,i'}; -N\epsilon_{i'}}^{(i')} w_{(i'),1}^{k_{1,i'}} \dots w_{(i'),m}^{k_{m,i'}} \right) \\
& \quad \left[ \sum_{k=(k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})} \left( c_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right) \right] \\
&= \left( \prod_{i=1}^n g_i(w_{(i),1}, \dots, w_{(i),m})^{-N\epsilon_i} \right) \cdot \left[ \sum_{k=(k_{j,i}) \in \text{Mat}_{m \times n}(\mathbb{Z})} \left( c'_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right) \right] \\
&= \left( \prod_{i=1}^n g_i^{-N\epsilon_i} \right) \cdot G = f(g_1, \dots, g_n).
\end{aligned}$$

□

### A.0.3 Regular singularity case

Suppose we have a multi-valued analytic function  $f(z_1, \dots, z_n)$  defined on  $(z_1, \dots, z_n) \in M = \Delta_{I_n, 0, \delta}^\times$  with a preferred single valued branch  $f^e(z_1, \dots, z_n)$  on  $M_0$  such that  $(z_1, \dots, z_n) = \delta$  is a regular singularity of  $f(z_1, \dots, z_n)$ . Therefore, there exist  $K, L \in \mathbb{N}$ , and  $\alpha_1, \dots, \alpha_K, a_{k_1, \dots, k_n; i_1, \dots, i_n} \in \mathbb{C}$  for any  $k_j \in D_j = \cup_{p=1}^K (\alpha_p + \epsilon_j \mathbb{N})$ ,

and  $i_j \in \{0, \dots, L\}$ ,  $j = 1, \dots, n$ , such that

$$f^e(z_1, \dots, z_n) = \sum_{\substack{k_j \in D_j, \\ j=1, \dots, n}} \sum_{i_1, \dots, i_n=0}^L a_{k_1, \dots, k_n; i_1, \dots, i_n} z_1^{k_1} \cdots z_n^{k_n} \log(z_1)^{i_1} \cdots \log(z_n)^{i_n}, \quad (\text{A.33})$$

where  $z_j^{k_j} = e^{k_j l_0(z_j)}$ ,  $\log(z_j) = l_0(z_j)$ . The right-hand side of (A.33) is absolutely convergent (and therefore single valued analytic) on the region  $(z_1, \dots, z_n) \in M_0$ .

The right-hand side of (A.33) can be written as

$$\begin{aligned} & \sum_{\substack{k_j \in \epsilon_j \mathbb{N}, \\ j=1, \dots, n}} \sum_{q_1, \dots, q_n=1}^K \sum_{i_1, \dots, i_n=0}^L a_{\alpha_{q_1}+k_1, \dots, \alpha_{q_n}+k_n; i_1, \dots, i_n} z_1^{\alpha_{q_1}+k_1} \cdots z_n^{\alpha_{q_n}+k_n} \log(z_1)^{i_1} \cdots \log(z_n)^{i_n} \\ = & \sum_{q_1, \dots, q_n=1}^K \sum_{i_1, \dots, i_n=0}^L (z_1^{\alpha_{q_1}} \cdots z_n^{\alpha_{q_n}} \log(z_1)^{i_1} \cdots \log(z_n)^{i_n}) \cdot f_{\alpha_{q_1}, \dots, \alpha_{q_n}; i_1, \dots, i_n}(z_1, \dots, z_n), \end{aligned} \quad (\text{A.34})$$

$$\text{where } f_{\alpha_{q_1}, \dots, \alpha_{q_n}; i_1, \dots, i_n}(z_1, \dots, z_n) = \sum_{\substack{k_j \in \epsilon_j \mathbb{N}, \\ j=1, \dots, n}} a_{\alpha_{q_1}+k_1, \dots, \alpha_{q_n}+k_n; i_1, \dots, i_n} z_1^{k_1} \cdots z_n^{k_n}.$$

Then,  $(z_1, \dots, z_n) = \delta$  is a pole for each  $f_{\alpha_{q_1}, \dots, \alpha_{q_n}; i_1, \dots, i_n}$ .

Suppose we have functions  $g_1, \dots, g_n$  satisfying the same conditions in section 2. Using corollary A.0.6, we have formal series for  $g_k(w_{(k),1}, \dots, w_{(k),m})^{\alpha_{q_k}}$ , and  $\log(g_k(w_{(k),1}, \dots, w_{(k),m}))$ . Using theorem A.0.7 (series (A.25)), we have formal series for each  $f_{\alpha_{q_1}, \dots, \alpha_{q_n}; i_1, \dots, i_n}(z_1, \dots, z_n)$ . Therefore, we have a formal series

$$\sum_{k=(k_j, i) \in \text{Mat}_{m \times n}(\mathbb{C})} \left( d_k \prod_{\substack{i=1, \dots, n \\ j=1, \dots, m}} w_{(i),j}^{k_{j,i}} \right) \quad (\text{A.35})$$

for  $f^e(g_1, \dots, g_n)$  using (A.34), where  $d_k \in \mathbb{C}$ .

**Theorem A.0.8.** *The formal series (A.35) is absolutely convergent to*

$$\begin{aligned}
 & \sum_{q_1, \dots, q_n=1}^K \sum_{i_1, \dots, i_n=0}^L (e^{\alpha_{q_1} l_{p_1}(g_1)} \dots e^{\alpha_{q_n} l_{p_n}(g_n)} l_{p_1}(g_1)^{i_1} \dots l_{p_n}(g_n)^{i_n}) \\
 & \qquad \qquad \qquad \cdot f_{\alpha_{q_1}, \dots, \alpha_{q_n}; i_1, \dots, i_n}(g_1, \dots, g_n) \\
 = & \sum_{\substack{k_j \in D_j, \\ j=1, \dots, n}} \sum_{i_1, \dots, i_n=0}^L a_{k_1, \dots, k_n; i_1, \dots, i_n} e^{k_1 l_{p_1}(g_1)} \dots e^{k_n l_{p_n}(g_n)} l_{p_1}(g_1)^{i_1} \dots l_{p_n}(g_n)^{i_n}, \quad (\text{A.36})
 \end{aligned}$$

where  $p_1, \dots, p_n \in \mathbb{Z}$  are defined in corollary A.0.6, on the same region in theorem A.0.7.

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